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LYAPUNOV EXPONENTS AND PROBLEMS OF LINEARIZATION. FROM STABILITY TO CHAOS

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Introduction

At the present time there exist two different approaches to the investigation of dynamic systems. The first of them is the classical theory of the stability of motion. There are many books devoted to this approach [1,5, 7-9,12,18,29,30,42,47]. In the last decades the most intensive work in this direction is observed in the problems of mechanics [17,31,40] and in the control theory [4,14,17,25, 31,37].

The second direction in the investigation of dynamic systems is the theory of chaos, to which the books [3,11,33, 35,39,43,44,46,48,49] are devoted.

In the present work an attempt is made to combine various elements of mathematical constructions, which had been used in the framework of one of two approximate methods mentioned.

It turned out to be that the definition of Lyapunov exponents via singular values has a significant advantage in comparison with the classical definition of Lyapunov exponents from the point of view of their estimation. This fact is demonstrated in Chapters 1 and 2 for continuous and discrete systems.

In Chapter 3 by methods, which usually are used for the estimation of dimensions of attractors, some new results are obtained in the classical problem of stability by the first approximation.

In Chapter 4 the methods, used previously to analyze the stability of discrete control systems, are developed for obtaining frequency conditions of existence of strange attractors.

This book is a natural continuation of the books [24,25]. In these books the authors describe the frequency methods to be used here in Chapters 1,2 and 4 in details. Therefore, the reader easily becomes familiar with the methods which are set forth here, if he is familiar with [24,25].

The main motivation to write this book was my opinion that all the methods demonstated here may be developed and generalized in many directions. Thus, the results obtained may be applied to various systems having stable or chaotic character.

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CHAPTER I

ESTIMATES OF LYAPUNOV EXPONENTS AND THE HAUSDORFF DIMENSION OF ATTRACTORS

Let us consider a system

$$\frac{dx}{dt} = A(t)x, \quad x \in \mathbf{R}^n,$$

where $A(t)$ a real $n \times n$-matrix. Let $X(t)$ be its fundamental matrix with initial data $X(0) = I$, where I is an unit matrix.

Further we will denote by $\mathbf{R}^n$ an Euclidean n-space and by $\mathbf{C}^n$ a unitary one.

Let us recall that the square roots of the eigenvalues of matrix $X(t)^*X(t)$ are called the singular numbers $\rho_1(t) \geq \geq \ldots \geq \rho_n(t)$ of the matrix $X(t)$. Let us also recall the following definition of Lyapunov exponents ν_j [24,25]:

$$\nu_j = \ln \varlimsup_{t \to +\infty} \rho_j(t)^{1/t}.$$

The estimates of the Lyapunov exponents, developed earlier in the framework of the classical theory of linear systems [1,7,12,30], recently obtained a new motivation: they are used to estimate the dimensions of attractors [24,25,46].

First, we estimate from above the value ν_1, generalizing the classical Wazewski inequality [1,8,12] and making more precise the estimates of solutions obtained earlier by the Gronwall lemma [8,12].

Let L be a linear operator in the space of continuous functions given on $[0, +\infty)$. By $(Lv)(t)$ denote the value of the function $Lv(\cdot)$ at the point t.

We now assume that L has the following property of monotonicity. If $v_1(t) \leq v_2(t), \quad \forall\, t \geq 0$, then $\big(Lv_1\big)(t) \leq \leq \big(Lv_2\big)(t), \ \forall\, t \geq 0$. We also suppose that if $v_1(t) \equiv v_2(t)$ on the interval $[0, \tau]$, then $\big(Lv_1\big)(\tau) = \big(Lv_2\big)(\tau)$.

Suppose that for all function $v(t) \geq 0, \ \forall\, t \geq 0$ and the number $C \geq 0$ the following inequality

$$v(t) \leq C + \big(Lv\big)(t), \qquad \forall\, t \geq 0 \tag{1}$$

is satisfied.

Lemma 1. *If for some number* $q \in (0, 1)$

$$\big(L1\big)(t) \leq q, \qquad \forall\, t \geq 0, \tag{2}$$

then

$$v(t) \le \frac{C}{1-q}, \qquad \forall\, t \ge 0. \tag{3}$$

Proof. Inequality (3) results from the following relation

$$\max_{[0,T]} v(t) \le C + q \max_{[0,T]} v(t),$$

which in turn results from (1) and (2).

We now suppose that for the continuous nonnegative functions $v(t)$, $\omega(t,\tau)$, the continuous function $p(t)$, and the number $C \ge 0$ we have

$$v(t) \le C e^{p(t)} + \int_o^t \omega(t,\tau) v(\tau)\, d\tau, \qquad \forall\, t \ge 0. \tag{4}$$

Lemma 2. *If for some number* $q \in (0,1)$

$$\int_0^t \omega(t,\tau) \exp\big(p(\tau) - p(t)\big)\, d\tau \le q, \qquad \forall\, t \ge 0, \tag{5}$$

then

$$v(t) \le \frac{C}{1-q} e^{p(t)}, \qquad \forall\, t \ge 0. \tag{6}$$

Proof. Let us write inequality (4) as follows

$$\big[e^{-p(t)} v(t)\big] \le C + \int_0^t \omega(t,\tau) e^{p(\tau)-p(t)} \big[e^{-p(\tau)} v(\tau)\big]\, d\tau.$$

Applying Lemma 1 to this inequality and noting that

$$(L1)(t) = \int_0^t \omega(t,\tau) e^{p(\tau)-p(t)} d\tau,$$

we get

$$\big[e^{-p(t)} v(t)\big] \le \frac{C}{1-q},$$

which proves the lemma.

Let us name one of the simplest applications of Lemma 2.

Consider the system

$$\dot{x}_i = a_i x_i + \sum_{j=1}^{n} b_{ij}(t) x_j, \tag{7}$$

where a_i are complex numbers; $b_{ij}(t)$ are complex-valued continuous functions , $i = 1, \dots , n,\ \ t \in \mathbf{R}^1$. We denote

$$\alpha_i = Rea_i, \qquad b_i(t) = \max_j \left| b_{ij}(t) \right|.$$

Theorem 1. *Let for certain numbers λ and $q \in (0,1)$ the inequalities $\lambda \geq \max\limits_i \alpha_i$,*

$$\int_0^t e^{\lambda(\tau - t)} \sum_{i=1}^{n} b_i(\tau) e^{\alpha_i(t-\tau)} d\tau \leq q, \quad \forall\ t \geq 0 \tag{8}$$

be true.

Then

$$\sum_{i=1}^{n} \left| x_i(t) \right| \leq \frac{e^{\lambda t}}{1-q} \sum_{i=1}^{n} \left| x_i(0) \right|, \quad \forall\ t \geq 0. \tag{9}$$

Proof. We write the solution of system (7) in the form:

$$x_i(t) = e^{a_i t} \left(x_i(0) + \int_0^t e^{-a_i \tau} \sum_{j=1}^{n} b_{ij}(\tau) x_j(\tau)\, d\tau \right).$$

Hence it follows that

$$\sum_{i=1}^{n} |x_i(t)| \leq e^{\lambda t} \sum_{i=1}^{n} |x_i(0)| +$$
$$+ \int_0^t \left(\sum_{i=1}^{n} b_i(\tau) e^{\alpha_i(t-\tau)} \right) \left(\sum_{j=1}^{n} |x_j(\tau)| \right) d\tau.$$

Thus, inequality (4) with $p(t) = \lambda t$,

$$v(t) = \sum_{i=1}^{n} \left| x_i(t) \right|,$$

$$C = \sum_{i=1}^{n} \left| x_i(0) \right|, \qquad \omega(t,\tau) = \sum_{i=1}^{n} b_i(\tau) e^{\alpha_i (t-\tau)}$$

is valid.

From (8) it follows that condition (5) is satisfied. Therefore, inequality (6) coincides with estimate (9).

We also denote $\beta_i = \sup\limits_{t \geq 0} b_i(t)$.

Corollary 1. *Suppose that for some number λ we have*

$$\lambda \geq \max_i \alpha_i, \qquad q := \sum_{i=1}^{n} \frac{\beta_i}{\lambda - \alpha_i} < 1.$$

Then estimate (9) is true.

It can be noted that in the case when for some i and j $\alpha_i \neq \alpha_j$ and $b_{ij}(t) \equiv b_{ik}(t)$, $\forall\, i, j, k$ the estimate of an exponential growth of the solutions of linear systems (9) is always more precise than the estimate obtained by the Gronwall lemma [8,12]. Besides we can easily see that estimate (9) is invariant with respect to nonsingular linear substitutions in system (7).

We now formulate a certain generalized Wazewski inequality [12,18].

Consider the system

$$\dot{x}_i = A_i(t) x_i + \sum_{j=1}^{n} B_{ij}(t) x_j, \tag{10}$$

where $A_i(t)$ are continuous $n_i \times n_i$–matrices, $B_{ij}(t)$ are continuous $n_i \times n_j$–matrices. By $\mu_i(t)$ we denote the maximal eigenvalue of the matrix $A_i(t) + A_i(t)^*$. Suppose that for the continuous nonnegative functions $b_i(t)$ the following estimate

$$2 \sum_{j=1}^{n} x_i^* B_{ij}(t) x_j \leq b_i(t) \sum_{j=1}^{n} |x_j|^2,$$
$$\forall\, x_i \in \mathbf{R}^{n_i}, \ \forall\, i = 1, \dots, n$$

is true. Let us also denote

$$p_i(t) = \int\limits_0^t \mu_i(\tau)\, d\tau.$$

Theorem 2. *If for a continuous function $p(t)$ such taht $p(t) \geq p_i(t)$, $\forall\, t \geq 0$ and a number $q \in (0,1)$*

$$\int\limits_0^t \exp\big(p(\tau) - p(t)\big) \sum_{i=1}^{n} b_i(\tau) \exp\big(p_i(t) - p_i(\tau)\big)\, d\tau \leq q, \tag{11}$$

$\forall\ t \geq 0,$

then

$$\sum_{i=1}^{n} \big|x_i(t)\big|^2 \leq \frac{e^{p(t)}}{1-q} \sum_{i=1}^{n} \big|x_i(0)\big|^2. \tag{12}$$

For $i = 1,\ b_1(t) \equiv 0$ we have the Wazewski inequality.

Proof. From equation (10) the following inequality

$$\big|x_i(t)\big|^2 \leq e^{p_i(t)} \left(\big|x_i(0)\big|^2 + \int\limits_0^t e^{-p_i(\tau)} b_i(\tau) \sum_{j=1}^{n} \big|x_j(\tau)\big|^2 d\tau \right)$$

can be obtained. Hence

$$\sum_{i=1}^{n} \big|x_i(t)\big|^2 \leq e^{p(t)} \sum_{i=1}^{n} \big|x_i(0)\big|^2 +$$
$$+ \int\limits_0^t \left(\sum_{i=1}^{n} e^{p_i(t) - p_i(\tau)} b_i(\tau) \right) \left(\sum_{j=1}^{n} \big|x_j(\tau)\big|^2 \right) d\tau.$$

Thus inequality (4) with

$$v(t) = \sum_{i=1}^{n} \big|x_i(t)\big|^2, \quad C = v(0),$$

$$\omega(t,\tau) = \sum_{i=1}^{n} b_i(\tau) e^{p_i(t) - p_i(\tau)}$$

is valid. From (11) using Lemma 2 we obtain estimate (12).

Example. It will be shown that Theorem 2 can give a more precise estimate than the Wazewski inequality. Consider the following system

$$\dot{x}_1 = 2x_1,$$
$$\dot{x}_2 = b(t)x_1,$$

where the continuous function $b(t)$ satisfies the condition $0<\gamma\le|b(t)| \le \beta < 4 \ \ \forall\, t \ge 0$. Here γ and β are some numbers. In this case we get $\mu_1 = 4,\ \mu_2 = 0$. Then for $p(t) = 4t$ condition (11) can be written in the form:

$$\frac{\beta}{4} < 1.$$

Thus, estimate (12) take the form

$$x_1(t)^2 + x_2(t)^2 \le \frac{e^{4t}}{1-\frac{\beta}{4}}\big(x_1(0)^2 + x_2(0)^2\big).$$

By the Wazewski inequality we get the following estimate

$$x_1(t)^2 + x_2(t)^2 \le \exp \int\limits_0^t \big(2+\sqrt{4+b(\tau)^2}\,\big)\,d\tau\, \big(x_1(0)^2+x_2(0)^2\big).$$

It is clear that for sufficiently large t the above estimate is less precise than estimate (12).

As an example, consider the following Lorenz system [11]

$$\dot{X} = -dX + dY, \quad \dot{Y} = (r-Z)X - Y,$$

$$\dot{Z} = -bZ + XY, \tag{13}$$

where d, r, b are positive numbers. Setting $X = \kappa\sigma$ we write the matrix of the system, linearized along a certain trajectory $\sigma(t),\ Y(t),\ Z(t)$:

$$A(t) = \begin{pmatrix} -d & d/\kappa & 0 \\ \kappa\big(r - Z(t)\big) & -1 & -\kappa\sigma(t) \\ \kappa Y(t) & \kappa\sigma(t) & -b \end{pmatrix}.$$

In order to obtain the estimates of the Hausdorff dimension of attractors and the global stability of system (13) it is important to estimate the maximal Lyapunov exponents of the linear system such that its matrix is as follows

$$-A(t)^* = \begin{pmatrix} d & -\kappa\big(r - Z(t)\big) & -\kappa Y(t) \\ -d/\kappa & 1 & -\kappa\sigma(t) \\ 0 & \kappa\sigma(t) & b \end{pmatrix}.$$

For this purpose this system is represented as (10) with

$$A_1(t) = d, \quad B_{11}(t) = 0,$$

$$B_{12}(t) = \big(-\kappa\big(r - Z(t)\big), -\kappa Y(t)\big),$$

$$A_2(t) = \begin{pmatrix} 1 & -\kappa\sigma(t) \\ \kappa\sigma(t) & 1 \end{pmatrix},$$

$$B_{21}(t) = \begin{pmatrix} -d/\kappa \\ 0 \end{pmatrix}, \quad B_{22}(t) = \begin{pmatrix} 0 & 0 \\ 0 & b-1 \end{pmatrix},$$

$$b_1(t) = \kappa\sqrt{\big(r - Z(t)\big)^2 + Y(t)^2},$$

$$b_2(t) = \max\left(\frac{d}{\kappa}, 2(b-1)\right).$$

Further, we suppose that $d/\kappa \geq 2(b-1)$. It is clear that $\mu_1(t) = 2d$, $\mu_2(t) = 2$. Therefore condition (11) for $p(t) = \lambda t$ takes the form:

$$\frac{d}{\kappa(\lambda-2)} + \kappa e^{(2d-\lambda)t} \int_0^t e^{(\lambda-2d)\tau} \sqrt{\big(r - Z(\tau)\big)^2 + Y(\tau)^2}\, d\tau \leq$$
$$\leq q.$$

Assume that on the attractor of system (13) we have

$$(\lambda - 2d)e^{(2d-\lambda)t} \int_0^t e^{(\lambda-2d)\tau} \sqrt{\big(r - Z(\tau)\big)^2 + Y(\tau)^2}\, d\tau \leq \\ \leq M, \qquad \forall\, t \geq 0. \tag{14}$$

Then, choosing

$$\kappa^2 = \frac{d(\lambda - 2d)}{(\lambda - 2)M},$$

we get the following estimate for λ:

$$\frac{4\,dM}{(\lambda-2)(\lambda-2d)} < 1. \tag{15}$$

It is easy to see that in inequality (15) we can consider only such λ for which the left side of the inequality is close to 1. In this case the inequality $d/\kappa \geq 2(b-1)$ is satisfied if $\lambda > 4b-2$.

From the above reasoning, the results of [24,25,46], and estimate (15) we have the following theorem.

Theorem 3. *If $d > b-2$ and inequality (15) holds for $\lambda = 2(d+b+1)/(1-S)$, then for the Hausdorff dimension $\dim K$ of the attractor K we have the estimate $\dim K \leq 2+S$. If this inequality is valid for $\lambda = 2(d+b+1)$, then any solution of system (13) tends to a certain equilibrium as $t \to +\infty$.*

It is well known [24,25] that on attractor of system (13) one can chosen the number M such that it satisfies the inequality $M \leq \gamma$, where

$$\gamma = \begin{cases} r & \text{for } \ b \leq 2, \\ \dfrac{br}{2\sqrt{b-1}} & \text{for } \ b \geq 2. \end{cases}$$

Note that if $M = \gamma$ and $1 < b < d$, then estimate (15) coincides with the well-known Eden-Foias-Temam estimate [15].

Sometimes estimate (9) can be made more precise, taking into account the peculiarities of the structure of the linear system. For example, let us consider a system

$$\begin{aligned} \dot{x}_{i1} &= a_{i1}x_{i1} + b_{i1}(t)\sigma, \\ \dot{x}_{i2} &= a_{i2}x_{i2} + b_{i2}(t)x_{i1}, \\ \dot{x}_{i3} &= a_{i3}x_{i3} + b_{i3}(t)x_{i2}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \dot{x}_{im(i)} &= a_{im(i)}x_{im(i)} + b_{im(i)}(t)x_{im(i)-1}, \end{aligned} \tag{16}$$

$$\sigma = \sum_{i=1}^{r} x_{im(i)},$$

where a_{ij} are complex numbers; $b_{ij}(t)$ are complex continuous functions. We now introduce

$$\alpha_{ij} = Re\, a_{ij}, \quad \beta_{ij} = \sup_{t \geq 0} \left| b_{ij}(t) \right|.$$

Theorem 4. *Let for some number λ the inequalities $\lambda > \max\limits_{ij} \alpha_{ij}$,*

$$Q := \sum_{i=1}^{r} \prod_{j=1}^{m(i)} \frac{\beta_{ij}}{\lambda - \alpha_{ij}} < 1. \tag{17}$$

be valid.

Then

$$\left|\sigma(t)\right| \le \frac{C}{1-Q} e^{\lambda t}, \qquad \forall\, t \ge 0, \tag{18}$$

where C is some constant depending on the initial data $x_{ij}(0)$ and the values β_{ij}, α_{ij}.

Proof. For system (16) we have the following simple estimate

$$\left|\sigma(t)\right| \le Ce^{\lambda t} + \sum_{i=1}^{r} \left|\widetilde{x}_{im(i)}(t)\right|, \qquad \forall\, t \ge 0, \tag{19}$$

where $\widetilde{x}_{ij}(t)$ are solutions of the corresponding scalar differential equations of system (16) with initial data $\widetilde{x}_{ij}(0) = 0$, and C is a certain sufficiently large number depending on $x_{ij}(0)$, α_{ij}, β_{ij}.

On the other hand, it is well known (for example, from the Wazewski inequality) that there exist the numbers $R > 0$ and $\gamma > \lambda$ for which the estimate

$$\left|\sigma(t)\right| \le Re^{\gamma t}, \qquad \forall\, t \ge 0 \tag{20}$$

is valid.

Taking into consideration the estimate from the equations of system (16), we shall sequentially get

$$\begin{aligned}
&\left|\widetilde{x}_{i1}(t)\right| \le e^{\alpha_{i1} t} \int_0^t Re^{(\gamma - \alpha_{i1})\tau} \beta_{i1}\, d\tau \le \frac{R\beta_{i1}}{\lambda - \alpha_{i1}} e^{\gamma t}, \\
&\left|\widetilde{x}_{i2}(t)\right| \le e^{\alpha_{i2} t} \int_0^t \frac{R\beta_{i1}}{\lambda - \alpha_{i1}} e^{(\gamma - \alpha_{i2})\tau} \beta_{i2}\, d\tau \le \\
&\le \frac{R\beta_{i2}}{(\lambda - \alpha_{i2})} \frac{\beta_{i1}}{(\lambda - \alpha_{i1})} e^{\gamma t}, \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&\left|\widetilde{x}_{im(i)}(t)\right| \le Re^{\gamma t} \prod_{j=1}^{m(i)} \frac{\beta_{ij}}{\lambda - \alpha_{ij}}.
\end{aligned} \tag{21}$$

Thus, from inequality (19) and above estimates one can derive

$$\left|\sigma(t)\right| \le Ce^{\lambda t} + QRe^{\gamma t}, \qquad \forall\, t \ge 0. \tag{22}$$

Let us now make of use estimate (22) instead of estimate (20) when considering equalities (16) and estimate (19). Then we obtain the following inequality

$$\left|\sigma(t)\right| \le Ce^{\lambda t}(1+Q) + Q^2 Re^{\gamma t}, \qquad \forall\, t \ge 0.$$

Repeating this procedure k times, we get the following estimate:

$$\left|\sigma(t)\right| \le Ce^{\lambda t}(1+Q+\ldots+Q^k) + Q^k Re^{\gamma t}, \qquad \forall\, t \ge 0.$$

Passing in this inequality to the limit as $k \to +\infty$ we obtain estimate (18).

Note also that the estimate of the exponential growth of solutions (18) is invariant with respect to linear transformations of the phase space.

Let us remark that in the case $\lambda < 0$ Theorems 1 and 4 are criteria of the stability of linear systems. Sometimes Theorem 4 gives better result than a well-known frequency domain circle criterion of stability [25]. It can be shown, for example, in the case: $r{=}1$, $\left|b_{1j}(t)\right|{\le}1$, $\forall\, t{\ge}0$, $\forall\, j$, $m(1) > 1$. Here estimate (18) gives the following condition of stability: $\alpha_{1j}{<}0$, $\forall\, j$, $|\alpha_{11}\alpha_{12}\ldots\alpha_{1m(1)}| > 1$. The circle criterion leads to this result in the case $b_{12}(t) \equiv$ $\equiv b_{13}(t) \equiv \ldots \equiv b_{1m(1)}(t) \equiv 1$ only.

Now consider the estimates of Lyapunov exponents ν_j in the case $j = 1, \ldots, n$. For this purpose we continue our study of the following system

$$\frac{dx}{dt} = A(t)x, \qquad x \in \mathbf{R}^n, \tag{23}$$

where $A(t)$ is a real continuous $n \times n$-matrix.

Denote by $u_1(t),\ldots,u_n(t)$ an orthonormal system of eigenvectors of the matrix $X(t)^*X(t)$ such that $X(t)^*X(t)u_j(t) =$ $= \rho_j(t)^2 u_j(t)$. Recall [16] that the linear operator $X(t)$ transforms the orthonormal system $u_1(t), \ldots, u_n(t)$ into the orthogonal one and $\left|X(t)u_j(t)\right| = \rho_j(t)$.

We now consider a certain linear k-dimensional subspace $L_k \subset \mathbf{R}^n$ and nonnegative function $\alpha(t)$.

Lemma 3. *Suppose that for any vector $v \in L_k$, the estimate $|X(t)v| \ge \alpha(t)|v|$ is valid; then $\rho_k(t) \ge \alpha(t)$.*

Proof. Let us fix a value of the parameter t and choose a vector $v \in L_k$ such that $v^* u_j(t) = 0, \ \forall\, j = 1, \dots, k-1$. It is obvious that there exists nonzero vector v satisfying these hypotheses. Assuming

$$v = \sum_{j=k}^{n} \alpha_j u_j(t), \qquad \alpha_j \in \mathbf{R}^1,$$

we find

$$\alpha(t)^2 |v|^2 \le \bigl|X(t)v\bigr|^2 = \sum_{j=k}^{n} \alpha_j^2 \rho_j(t)^2 \le \rho_k(t)^2 |v|^2.$$

Lemma 3 is proved.

We now consider a symmetrical matrix H, which has no less than k negative eigenvalues, a quadratic form $V(x) = x^* H x$, and the set $\Omega = \bigl\{x \mid x^* H x < 0\bigr\}$.

Lemma 4. *Suppose that for some function $\lambda(t)$ and any vector $z \in \Omega$ the following inequality*

$$z^* H A(t) z + \lambda(t) z^* H z \le 0, \quad \forall\, t \ge 0 \tag{24}$$

is valid.

Then there exists a positive number β such that

$$\rho_k(t) \ge \beta \exp\left(-\int_0^t \lambda(\tau)\, d\tau\right), \quad \forall\, t \ge 0. \tag{25}$$

Proof. Let us rewrite estimate (24) as follows:

$$\left(V\bigl(x(t)\bigr) \exp 2 \int_0^t \lambda(\tau)\, d\tau\right)^{\bullet} \le 0.$$

Hence we find that $V\bigl(x(t)\bigr) \le V\bigl(x(0)\bigr) \exp\left(-2 \int_0^t \lambda(\tau)\, d\tau\right)$. From here and the condition on the spectrum of the matrix H we obtain that there exist k-dimensional hyperplane L_k and the number β such that for any $x(0) \in L_k$ we have the estimate $|x(t)| \ge \beta |x(0)| \exp\bigl(-\int_0^t \lambda(\tau) d\tau\bigr)$. This estimate and Lemma 3 prove Lemma 4. The following assertion is proved in the same way.

Lemma 5. *If for any vector $v \in L_k$ the following estimate $\left|X(t)v\right| \leq \alpha(t)|v|$ is true, then $\rho_{n-k+1}(t) \leq \alpha(t)$.*

Consider now the symmetrical matrix M having no less than k positive eigenvalues, the quadratic form $W(x) = x^* M x$, and the set $\Phi = \{x |\, x^* M x > 0\}$.

Lemma 6. *Suppose that for some function $\psi(t)$ and any vector $z \in \Phi$ we have*

$$z^* M A(t) z + \psi(t) z^* M z \leq 0, \quad \forall\, t \geq 0. \tag{26}$$

Then there exists a positive number C such that

$$\rho_{n-k+1}(t) \leq C \exp\left(-\int_0^t \psi(\tau)\, d\tau\right), \quad \forall\, t \geq 0. \tag{27}$$

Proof. Consider the system

$$\dot{y} = -A(t)^* y$$

and its fundamental matrix $Y(t)$ with initial data $Y(0) = I$. It is well known [12] that

$$Y(t)^* Y(t) = \left(X(t)^* X(t)\right)^{-1}.$$

Therefore $\gamma_j(t) = \rho_{n-j+1}(t)^{-1}$, where $\gamma_j(t)$ are singular numbers of the matrix $Y(t)$ such that $\gamma_1(t) \geq \ldots \geq \gamma_n(t)$.

We now introduce the following notations: $H = -M$, $\lambda(t) = -\psi(t)$, $A_0(t) = -A(t)^*$, $C = \beta^{-1}$. Then condition (26) takes the same form as (24) with $A(t) = A_0(t)$. It follows from Lemma 4 that

$$\gamma_k(t) \geq \beta \exp \int_0^t \psi(\tau)\, d\tau, \quad \forall\, t \geq 0.$$

Thus,

$$\rho_{n-k+1}(t) = \gamma_k(t)^{-1} \leq C \exp\left(-\int_0^t \psi(\tau)\, d\tau\right), \quad \forall\, t \geq 0.$$

Consider system (23) with $A(t) = A + BU(t)$, where A is a constant $n \times n$-matrix, B is a constant $n \times m$-matrix, $U(t)$ is a continuous $m \times n$-matrix.

Let us introduce the Hermitian forms $F_k(z,\xi)$ of the complex vector variables $z \in \mathbf{C}^n,\ \xi \in \mathbf{C}^m,\ \ k = 1,\dots,n$.

Assume that there exist the $n \times m$-matrices Q_k and the positive number ε such that the following inequalities

$$F_k(x, Q_k^* x) \geq \varepsilon |Q_k^* x|^2, \qquad \forall\ x \in \mathbf{R}^n \tag{28}$$

are satisfied. We also suppose that

$$F_k\big(x, U(t)x\big) \geq 0, \qquad \forall\ x \in \mathbf{R}^n, \quad \forall\ t \geq 0. \tag{29}$$

Theorem 5. *Let the pair (A,B) be controllable, the pairs (A,Q_k) be observable and for some set of the numbers $\lambda_1 \leq \ldots \leq \lambda_n$ the following hypotheses be satisfied:*

1) the matrix $A + BQ_k^* + \lambda_k I$ has k or more eigenvalues with positive real parts;

2) for all $k = 1,\dots,n$ and all $\omega \in \mathbf{R}^1$ we have

$$F_k\big(\big[(i\omega - \lambda_k)I - A\big]^{-1} B\xi, \xi\big) \leq 0, \qquad \forall\ \xi \in \mathbf{C}^m. \tag{30}$$

Then for the singular numbers $\rho_k(t)$ of the fundamental matrix of system (23) *there exist numbers $\beta_k > 0$ such that*

$$\rho_k(t) \geq \beta_k e^{-\lambda_k t}, \qquad \forall\ t \geq 0,\ \ k = 1,\dots,n. \tag{31}$$

Proof. From the hypothesis 2) of the theorem by the Yakubovich-Kalman lemma [24,25] it can be found the symmetrical matrix H_k for which the following inequality

$$\begin{gathered} 2z^* H_k\big[(A + \lambda_k I)z + B\xi\big] + F_k(z,\xi) \leq 0, \\ \forall\ z \in \mathbf{R}^n,\ \forall\ \xi \in \mathbf{R}^m. \end{gathered} \tag{32}$$

is true. Setting in (32) $\xi = Q^* x$ and taking into account (28), we obtain

$$2z^* H_k(A + \lambda_k I + BQ_k^*)z \leq -\varepsilon |Q_k^* z|^2, \qquad \forall\ z \in \mathbf{R}. \tag{33}$$

From inequality (33) and the hypothesis 1) of the theorem it follows that the matrix H_k has no less than k negative eigenvalues [24,25].

From relations (29) and (32) we concluded that for the function $V_k(x) = x^* H_k x$ and any solution $x(t)$ of system (23) we have

$$\begin{aligned}&\dot V_k\big(x(t)\big) + 2\lambda_k V_k\big(x(t)\big) = 2x(t)^* H_k\big[(A+\lambda_k I)x(t)+\\ &+BU(t)x(t)\big] + F_k\big(x(t), U(t)x(t)\big) -\\ &- F_k\big(x(t), U(t)x(t)\big) \le 0, \qquad \forall\, t \ge 0.\end{aligned}$$

This inequality and Lemma 4 result in the assertion of Theorem 5.

By the same proof and Lemma 6 we can obtain the following result.

Theorem 6. *Let the pair (A, B) be controllable, the pairs (A, Q_k) be observable and for some set of numbers $\lambda_1 \ge \ldots \ge \lambda_n$ the following hypotheses be satisfied:*

1) the matrix $A+BQ_k+\lambda_k I$ has k or more eigenvalues with negative real parts;

2) for all $k = 1, \ldots, n$ and all $\omega \in \mathbf{R}^1$ we have

$$F_k\big([(i\omega - \lambda_k)I - A]^{-1}B\xi,\ \xi\big) \le 0, \quad \forall\, \xi \in \mathbf{C}^m. \tag{34}$$

Then for the singular numbers $\rho_k(t)$ of the fundamental matrix of system (23) *there exist positive numbers β_k such that*

$$\rho_{n-k+1}(t) \le \beta_k e^{-\lambda_k t}, \quad \forall\, t \ge 0. \tag{35}$$

Let us now consider system (23) with $m = 1$ and $U(t) = u(t)C^*$, where C is a constant n-vector. Introduce the transfer function of system (23) $W(p) = C^*(A-pI)^{-1}B,\ \ p \in \mathbf{C}^1$. Further, we assume that the function $W(p)$ is nondegenerate [24,25] and the function $u(t)$ satisfies the inequalities $0 \le u(t) \le \mu, \ \ \forall t \ge 0$. Let us recall that $W(p)$ is nondegenerate if it has the denumeretor of degree n and the numeretor has no common zeroes with its denominator. The nondegeneracy is equivalent to controllability and observability.

Theorem 7. *Let for some set of numbers $\lambda_1 \le \ldots \le \lambda_n$ the following hypotheses be satisfied:*

1) the matrix $A+\lambda_k I$ has k or more eigenvalues with positive real parts;

2) for all $k = 1, \ldots, n$ and all $\omega \ge 0$ the inequality $\mu^{-1} + Re\, W(i\omega - \lambda_k) \ge 0$ is valid.

Then the assertion of Theorem 5 is true.

To prove of Theorem 7 it is sufficient to make use of Theorem 5 with the following Hermitian form

$$F_k(z,\xi) = \xi^*(C^*z - \mu^{-1}\xi) + (C^*z - \mu^{-1}\xi)^*\xi \tag{36}$$

and take $Q_k = \delta C$, where δ is a sufficiently small positive number. The controllability of (A, B) and observability of (A, Q_k) result from nondegeneracy of $W(p)$.

Remark 1. Hypothesis 2) of Theorem 2 can be easily written in more generalized form of circle criterion [24,25].

Remark 2. Let us recall the definition of Lyapunov exponents ν_j

$$\nu_j = \ln \overline{\lim_{t\to+\infty}} \rho_j(t)^{1/t}.$$

By this notation estimate (31) takes a very simple form: $\nu_j \geq -\lambda_j$.

Remark 3. The hypotheses of Theorem 7 are always satisfied for sufficiently large λ_n. Therefore from Theorem 7 we can always find estimate (31) for $\rho_n(t)$. On the other hand, for sufficiently small μ the hypotheses of Theorem 7 are satisfied for λ_k, which are close to $-\mathrm{Re}\,\kappa_k$. Here κ_k are eigenvalues of the matrix A such that $\mathrm{Re}\,\kappa_1 \geq \ldots \geq \mathrm{Re}\,\kappa_n$.

The same considerations can be made by Theorem 6 for the upper estimates ν_j.

Let us consider a system

$$\frac{dx}{dt} = Ax + Bf(x), \qquad x \in \mathbf{R}^n, \tag{37}$$

where A and B are constant $n \times n$ and $n \times m$-matrices, $f(x)$ is a continuously differentiable vector function.

Denote $\gamma(x) = \mathrm{Tr}\left(A + B\frac{\partial f}{\partial x}\right)$. Here $\frac{\partial f}{\partial x}$ is a Jacobi matrix of the vector-function $f(x)$ at the point x.

We now assume that the compact set K is invariant with respect to the shift operator along the trajectories of system (37), i.e., $x(t, K) = K$, $\forall\, t \in \mathbf{R}^1$. By $\dim K$ we denote the Hausdorff dimension K.

Let us suppose that condition (28) is satisfied and instead of inequality (29) we have the following relation:

$$F_k\left(z, \frac{\partial f}{\partial x} z\right) \geq 0, \qquad \forall\, z \in \mathbf{R}^n, \quad \forall\, x \in K. \tag{38}$$

Theorem 8. *Let the pair (A, B) be controllable, the pairs (A, Q_k) be observable and for a set of the numbers $\lambda_1 \leq \ldots \leq \lambda_n$ the hypotheses 1) and 2) of Theorem 5 be satisfied.*

Let for some natural number m and for $s \in [0,1]$ the following inequality

$$\gamma(x) + (1-s)\lambda_{m+1} + \sum_{j=m+2}^{n} \lambda_j < 0, \qquad \forall\, x \in K \tag{39}$$

be also valid. Then $\dim K < m + s$.

Theorem 9. *Let the pair (A,B) be controllable, the pairs (A,Q_k) be observable and for some set of numbers $\lambda_1 \geq \ldots \geq \lambda_n$ the hypotheses 1) and 2) of Theorem 6 be satisfied.*

Let for some natural number m and for $s \in [0,1]$ the following inequality

$$\sum_{j=n-m+1}^{n} \lambda_j + s\lambda_{n-m} > 0 \tag{40}$$

be true.
Then $\dim K < m + s$.

Suppose that system (37) is the Levinson dissipative one [12]. Suppose also that its domain of dissipativity Ω is generic in $\mathbf{C}^1$-topology and it is a bounded simply connected open set in $\mathbf{R}^n$. In other words, there exists a positive number ε such that for any continuously differentiable vector-function $g(x)$, satisfying the following condition

$$\big|f(x) - g(x)\big| + \left|\frac{\partial f}{\partial x} - \frac{\partial g}{\partial x}\right| \leq \varepsilon, \qquad \forall\, x \in \mathbf{R}^n,$$

and for any $x_0 \in \mathbf{R}^n$ a number $T(x_0, g)$ can be found, for which $x(t,x_0) \in \Omega,\ \forall\, t \geq T(x_0,g)$. Here $x(t,x_0)$ is a solution of the system $\dot{x} = Ax + Bg(x)$ with initial data $x(0,x_0) = x_0$.

Theorem 10. *Let all hypotheses of Theorem 5 and condition (38) for any $z \in \mathbf{R}^n$ and for any $x \in \Omega$, and condition (39) for any $x \in \Omega$ and for $m = 2,\ s = 0$ be satisfied.*

Then any solution of system (37) tends to a stationary set as $t \to +\infty$.

Theorem 11. *Let all hypotheses of Theorem 6, condition (38) for any $z \in \mathbf{R}^n$ and for any $x \in \Omega$, and condition (40) for $m = 2,\ s = 0$ be satisfied.*

Then any solution of system (37) tends to a stationary set as $t \to +\infty$.

The following estimate

$$\rho_1(t)\cdots\rho_d(t)\rho_{d+1}^s(t)\leq\beta\exp\left\{\int_0^t\left[\gamma(x(\tau))+\right.\right.$$
$$\left.\left.+(1-s)\lambda_{d+1}+\sum_{j=d+2}^{n}\lambda_j\right]d\tau\right\},$$

which results from Theorem 5 is the basis for the proof of Theorems 8 and 10. Here β is some constant, $\rho_j(t)$ are singular numbers of the fundamental matrices of linear systems, obtained as a result of the linearization of system (37) along its solutions $x(t)$.

Further, the proofs of Theorems 8 and 10 are the same as those given in [23–25].

Theorems 9 and 11 are proved in a similar way.

Let in system (37) be $m=1$, $f(x)=\varphi(C^*x)$, where C is a constant n-vector, and the following inequality

$$0\leq\frac{d\varphi}{d\sigma}\leq\mu,\qquad\forall\,\sigma\in\{\sigma=r^*x,\ x\in K\}\tag{41}$$

be valid.

Theorem 12. *Let all hypotheses of Theorem 7 be satisfied.*

If inequalities (39) *and* (41) *are valid, then* $\dim K < m+s$.

If the simply-connected domain of dissipativity Ω *is generic in* $\mathbf{C}^1$*-topology and inequalities* (39), (41) *hold for all* $x\in\Omega$ *and for* $m=2$, $s=0$, *then any solution of system (37) tends to the stationary set as* $t\to+\infty$.

This theorem as well as Theorem 7 is deduced from Theorems 8 and 10 by the Hermitian form (36).

Theorem 12 generalizes the results of R.Smith [45] and gives us more precise estimates. Note that recently years for analysing the global stability the method of using the compound matrices has been developed [28,32].

As an example of application of the theorems obtained, let us consider the Lorenz system

$$\dot X=-d(X-Y),\quad\dot Y=(r-Z)X-Y,\quad\dot Z=-bZ+XY\tag{42}$$

and some of its generalizations. Here $d,\ r,\ b$ are positive numbers.

We now write system (42) in the form of (37), where

$$A=\begin{pmatrix}-d&d&0\\0&-1&0\\0&0&-b\end{pmatrix},\quad B=\begin{pmatrix}0&0\\1&0\\0&1\end{pmatrix},$$

$$f(x) = \begin{pmatrix} (r-Z)X \\ XY \end{pmatrix}.$$

Let us construct the Hermitian form $F_k(z,\xi)$ as follows: $F_k(z,\xi) = \bar{\xi}_1 z_2 + \bar{\xi}_2 z_3 + \xi_1 \bar{z}_2 + \xi_2 \bar{z}_3 + M\big(\kappa|z_1|^2 + \kappa^{-1}|z_2|^2 + {}+\kappa^{-1}|z_3|^2\big)$. Here ξ_j and z_j are components of the vectors ξ and z, the positive numbers M and κ will be chosen later on.

In the case under consideration inequality (38) takes the form: $2\big[(r-Z)z_2 + Yz_3\big]z_1 + M\big(\kappa|z_1|^2 + \kappa^{-1}|z_2|^2 + {}+\kappa^{-1}|z_3|^2\big) \geq 0$. It can be easily seen that this condition is satisfied for all $z \in \mathbf{R}^3$ and for the following inequality:

$$(r-Z)^2 + Y^2 \leq M^2. \tag{43}$$

It is well known [24,25] that a certain domain of dissipativity Ω and, therefore, any invariant compact set K of the Lorenz system are situated in the set, defined by inequality (43) with

$$M = \begin{cases} r & \text{for} \quad b \leq 2, \\ \dfrac{br}{2\sqrt{b-1}} & \text{for} \quad b \geq 2. \end{cases} \tag{44}$$

Thus, if M is defined by equality (44), then condition (38) is satisfied.

In our case we have

$$[pI - A]^{-1}B\xi =$$
$$= \left(\frac{d}{(p+d)(p+1)}\xi_1, \frac{1}{(p+1)}\xi_1, \frac{1}{(p+b)}\xi_2\right)^T, \quad p \in \mathbf{C}^1.$$

Here the notation $(\ )^T$ is an operation of transposition. Therefore condition (30) is as follows:

$$2(\lambda_k - 1) \geq \kappa^{-1}M + \frac{\kappa d^2 M}{(\lambda_k - d)^2}, \qquad 2(\lambda_k - b) \geq \kappa^{-1}M.$$

If we take here $k = 3$, $\kappa = d^{-1}(\lambda_3 - d)$ and write inequality(39) in the form: $\lambda_3(1-s) < d + b + 1$, then we finally obtain the following assertion.

Corollary 2. *If $d \geq b - 2$ and for some $s \in [0,1]$*

$$\left(\frac{d+b+1}{1-s} - 1\right)\left(\frac{d+b+1}{1-s} - d\right) > Md, \tag{45}$$

then $\dim K < 2+s$. *If* (45) *holds for* $s = 0$, *then any solution of system* (42) *tends to a certain equilibrium as* $t \to +\infty$.

It can be remarked that from Corollary 2 we can find a well-known estimate of Eden-Foias-Temam [11]. Note that earlier by using the other methods we have obtained a similar result (see Theorem 3). In our opinion, the advantage of the "frequency" approach is that it is "almost automatically" transferred to more complicated equations than system (42). Let us consider, for example, the following system [44]:

$$\begin{aligned}
&\dot{X}_1 = \sigma_1(Y_1 - X_1), \quad \dot{Y}_1 = -Y_1 + X_1(r_1 - Z),\\
&\dot{Z} = -bZ + X_1Y_1 + X_2Y_2,\\
&\dot{X}_2 = \sigma_2(Y_2 - X_2), \quad \dot{Y}_2 = -Y_2 + X_2(r_2 - Z),
\end{aligned} \tag{46}$$

where $r_1,\ r_2,\ b,\ \sigma_1,\ \sigma_2$ are positive numbers. Without loss of generality, we shall assume that $r_2 \geq r_1$ and denote: $\alpha_0 = \min(1, \sigma_1, b)$, $L = r_2$ for $b \leq 2\alpha_0$, $L = r_2 b\big(4\alpha_0(b - \alpha_0)\big)^{-1/2}$ for $b \geq 2\alpha_0$,

$$V(x) = Y_1^2 + Y_2^2 + \frac{r_2 - r_1}{\sigma_1} X_1^2 + (r_2 - Z)^2.$$

Lemma 7. *There exists a domain of dissipativity* Ω *for system (46) such that* $\Omega \subset \big\{V(x) \leq L^2\big\}$.

The proof of the lemma is based on the consideration of the inequality $\dot{V}\big(x(t)\big) + 2\alpha V\big(x(t)\big) \leq \frac{1}{2} r_2^2 b^2 (b-\alpha)^{-1}$ for $\alpha \in [0, \alpha_0]$.

Write system (46) in the form:

$$A = \begin{pmatrix} -\sigma_1 & \sigma_1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0\\ 0 & 0 & -b & 0 & 0\\ 0 & 0 & 0 & -\sigma_2 & \sigma_2\\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1 \end{pmatrix},$$

$$f(x) = \begin{pmatrix} X_1(r_1 - Z)\\ X_1Y_1 + X_2Y_2\\ X_2(r_2 - Z) \end{pmatrix}.$$

Let us construct the Hermitian form $F_k(z, \xi)$ in the following way: $F_k(z, \xi) = \bar{\xi}_1 z_2 + \bar{\xi}_2 z_3 + \bar{\xi}_3 z_5 + \xi_1 \bar{z}_2 + \xi_2 \bar{z}_3 + \xi_3 \bar{z}_5 +$

$+L_1\left(\kappa_1|z_1|^2+\kappa_1^{-1}|z_2|^2+(\kappa_1^{-1}+\kappa_2^{-1})|z_3|^2+\kappa_2|z_4|^2+\kappa_2^{-1}|z_5|^2\right)$. Here $L_1 = L + r_2 - r_1$. The parameters κ_1 and κ_2 will be defined later on.

In our case inequality (38) will be satisfied, if

$$(r_1 - Z)^2 + Y_1^2 \leq L_1^2, \qquad (r_2 - Z)^2 + Y_2^2 \leq L_1^2. \tag{47}$$

It follows from Lemma 7 that these inequalities are valid in the set $\Omega \subset \{V(x) \leq L^2\}$. Therefore, condition (38) is satisfied.

Here condition (30) is equivalent to the inequalities

$$-2\mathrm{Re}\frac{1}{i\omega + 1 - \lambda_k} \geq \frac{L_1}{|i\omega + 1 - \lambda_k|^2}\left(\frac{1}{\kappa_1} + \frac{\kappa_1\sigma_1^2}{|i\omega+\sigma_1 - \lambda_k|^2}\right),$$

$$-2\mathrm{Re}\frac{1}{i\omega + 1 - \lambda_k} \geq \frac{L_1}{|i\omega + 1 - \lambda_k|^2}\left(\frac{1}{\kappa_2} + \frac{\kappa_2\sigma_2^2}{|i\omega+\sigma_2-\lambda_k|^2}\right),$$

$$-2\mathrm{Re}\frac{1}{i\omega + b - \lambda_k} \geq L_1\left(\kappa_1^{-1} + \kappa_2^{-1}\right)\frac{1}{|i\omega + b - \lambda_k|^2}.$$

Setting $\kappa_i = \sigma_i^{-1}|\lambda_k - \sigma_i|$, we shall write these conditions as follows:

$$\lambda_k - 1 \geq \frac{L_1\sigma_i}{|\lambda_k - \sigma_i|},$$

$$2(\lambda_k - b) \geq L_1\left(\frac{\sigma_1}{|\lambda_k - \sigma_1|} + \frac{\sigma_2}{|\lambda_k - \sigma_2|}\right). \tag{48}$$

Corollary 3. *Let* $\min(\sigma_1, \sigma_2) > \max(1, b)$ *and there exist numbers* $\lambda_4 \in (\sigma_1, \sigma_2)$ *and* $\lambda_5 > \max(\sigma_1, \sigma_2)$ *such that inequalities* (48) *and* $\sigma_1 + \sigma_2 + b + 2 > \lambda_5 + (1 - s)\lambda_4$, $s \in [-1, 1]$ *are valid.*

Then for any invariant compact set K *of system* (46) *we have* $\dim K < 3 + s$.

We shall formulate a conjecture which is a certain natural generalization of the results obtained.

Let us consider a system

$$\dot{x} = G(x), \qquad x \in \mathbf{R}^n \tag{49}$$

and denote by $\kappa_j(x)$ the eigenvalues of the matrix $\frac{\partial G}{\partial x}$, assuming that $\mathrm{Re}\,\kappa_1(x) \geq \ldots \geq \mathrm{Re}\,\kappa_n(x)$.

Conjecture. If for some $s \in [0, 1]$

$$\mathrm{Re}\,\kappa_1(x) + \ldots + \mathrm{Re}\,\kappa_d(x) + s\,\mathrm{Re}\,\kappa_{d+1}(x) < 0, \qquad \forall\, x \in K,$$

then $\dim K < d + s$, where K is a certain invariant compact set of system (49).

CHAPTER II

ESTIMATES OF LYAPUNOV EXPONENTS FOR DISCRETE SYSTEMS

Consider a system

$$x(k+1) = A(k)x(k), \qquad x \in \mathbf{R}^n, \ k \in \mathbf{Z} \tag{1}$$

and its fundamental matrix

$$X(k) = \prod_{j=0}^{k} A(j).$$

Here $A(j)$ is a real $n \times n$-matrix. Further, we shall use the following notations: $\mathbf{Z}$ is a set of integer numbers, $\mathbf{R}^n$ is an Eucledean n-dimensional space, and $\mathbf{C}^n$ is a unitary n-dimensional space.

Recall also the following definition of the Lyapunov exponent ν_j:

$$\nu_j = \ln \overline{\lim_{k \to +\infty}} \rho_j(k)^{1/k}.$$

In the present chapter the estimates of the Lyapunov exponents for continuous linear systems, obtained in previous chapter, are extended to discrete systems.

Let $u_1(k), \ldots, u_n(k)$ be an orthonormal system of eigenvectors ot the matrix $X(k)^*X(k)$ such that

$$X(k)^*X(k)u_j(k) = \rho_j(k)^2 u_j(k).$$

Recall that a linear operator $X(k)$ transforms the orthogonal system $u_1(k), \ldots, u_n(k)$ into the orthonormal one and $\left|X(k)u_j(k)\right| = \rho_j(k)$.

Let us introduce some m-dimensional linear subspace $L_m \subset \mathbf{R}^n$ and a nonnegative function $\alpha(k)$.

Lemma 1. *Let for any vector $v \in L_m$ an estimate $\left|X(k)v\right| \geq \alpha(k)|v|$ be satisfied. Then $\rho_m(k) \geq \alpha(k)$.*

Proof. Fix a number k and choose a vector $v \in L_m$ such that $v^* u_j(k) = 0, \ \forall\, j = 1, \ldots, m-1$. It is clear that there exists a nonzero vector v satisfying these conditions.

Having written the following expansion:

$$v = \sum_{j=m}^{n} \alpha_j u_j(k), \quad \alpha_j \in \mathbf{R}^1,$$

we obtain

$$\alpha(k)^2 |v|^2 \leq \bigl|X(k)v\bigr|^2 = \sum_{j=m}^{n} \alpha_j^2 \rho_j(k)^2 \leq \rho_m(k)^2 |v|^2$$

which proves the Lemma 1.

Consider now a symmetric matrix H having no less than m negative eigenvalues, a quadratic form $V(x) = x^* H x$, and a set $\Omega = \bigl\{x \mid x^* H x < 0\bigr\}$.

Lemma 2. *Let for some positive number λ and an arbitrary vector $z \in \Omega$ the inequality*

$$\frac{1}{\lambda^2}\, z^* A(k)^* H A(k) z \leq z^* H z \tag{2}$$

be satisfied. Then there exists a number $\beta > 0$ such that

$$\rho_m(k) \geq \beta \lambda^k, \quad \forall\ k \in \mathbf{Z},\ k > 0. \tag{3}$$

Proof. Write estimate (2) in the form $V\bigl(x(k+1)\bigr) \leq$ $\leq \lambda^2 V\bigl(x(k)\bigr)$. From here it follows that an inequality $V\bigl(x(k)\bigr) \leq V\bigl(x(0)\bigr)\lambda^{2k}$ holds. Then from the conditions for the spectrum of the matrix H we can deduce the existence of an m-dimensional hyperplane L_m and the existence of a number $\beta > 0$ such that for any $x(0) \in L_m$ the following estimates hold:

$$V\bigl(x(0)\bigr) < 0, \quad \bigl|x(k)\bigr| \geq \beta \bigl|x(0)\bigr| \lambda^k.$$

Using this estimate and Lemma 1, we obtain the statement of Lemma 2.

Consider now a symmetric matrix M with no less than m positive eigenvalues.

Lemma 3. *Let* $\det A(k) \neq 0, \quad \forall\, k \in \mathbf{Z}$, *and for some positive number* θ *and an arbitrary vector* $z \in \mathbf{R}^n$ *the following inequality be satisfied:*

$$\frac{1}{\theta^2}\, z^* A(k) M A(k)^* z \leq z^* M z. \tag{4}$$

Then there exists a number γ *such that*

$$\rho_{n-m+1}(k) \leq \gamma \theta^k, \quad \forall\, k \in \mathbf{Z},\ k > 0. \tag{5}$$

Proof. Together with system (1), consider a system

$$y(k+1) = \left[A(k)^{-1}\right]^* y(k). \tag{6}$$

It is obvious that for a fundamental matrix $Y(k)$ of this system an equality

$$X(k) Y(k)^* = I$$

is satisfied. From here it follows that

$$X(k)^* X(k) = \left[Y(k)^* Y(k)\right]^{-1}.$$

Therefore, for singular numbers $\kappa_j(k)$ of a matrix $Y(k)$ such that $\kappa_1(k) \geq \ldots \geq \kappa_n(k)$, an equality $\kappa_j(k) = \rho_{n-j+1}(k)^{-1}$ is satisfied.

Rewrite now inequality (4) in the following form:

$$\theta^2 y^* \left[A(k)^{-1}\right] H \left[A(k)^{-1}\right]^* y \leq y^* H y, \quad \forall\, y \in \mathbf{R}^n.$$

Here $y = A(k)^* z, \quad H = -M, \quad \lambda = \theta^{-1}$. Thus, for system (6) all conditions of Lemma 2 are satisfied and consequently for some $\beta > 0$ the following inequality holds:

$$\kappa_m(k) \geq \beta \theta^{-k}, \quad \forall\, k \in \mathbf{Z},\ k > 0.$$

From here it follows that

$$\rho_{n-m+1}(k) = \kappa_m(k)^{-1} \leq \frac{1}{\beta}\, \theta^k, \quad \forall\, k \in \mathbf{Z},\ k > 0.$$

Consider system (1) with

$$A(k) = A + B U(k) C^*,$$

where A, B and C are constant matrices $n\times n$, $n\times l$, $n\times r$ respectively, $U(k)$ is an $l\times r$-matrix.

Let us introduce a transfer matrix

$$W(p)=C^*(A-pI)^{-1}B,\quad p\in \mathbf{C}^1$$

and a Hermitian form $F_m(\sigma,\xi)$ of complex vector variables $\sigma\in\mathbf{C}^r$, $\xi\in\mathbf{C}^l$.

We shall assume that

$$F_m(\sigma,0)\geq 0,\quad \forall\ \sigma\in\mathbf{R}^r. \tag{7}$$

Let also an inequality

$$F_m\big(\sigma,U(k)\sigma\big)\geq 0,\quad \forall\ \sigma\in\mathbf{R}^r,\ \forall\ k\in\mathbf{Z},\ k>0 \tag{8}$$

be satisfied.

Theorem 1. *Let for some set of positive numbers $\lambda_1\geq\ldots\geq\lambda_n$ the following condition be satisfied:*

1) a matrix $\lambda_m^{-1}A$ has m or more eigenvalues outside of the circle of radius 1 with its centre at zero point,

2) for all values $p\in\mathbf{C}^1$ such that $|p|=1$ the following inequality is true:

$$F_m\big(-W(\lambda_m p)\xi,\xi\big)<0,\quad \forall\ \xi\in\mathbf{C}^l,\ \xi\neq 0. \tag{9}$$

Then for singular values $\rho_m(k)$ of the fundamental matrix $X(k)$ of system (1) there exists a positive number β_m such that

$$\rho_m(k)\geq\beta_m\lambda_m^k,\quad \forall\ k\in\mathbf{Z},\ k>0. \tag{10}$$

Proof. From the condition 2) of the theorem, according to the Kalman-Szego lemma [37], we see that there exists a real matrix H_m such that

$$\begin{gathered}\frac{1}{\lambda_m^2}[Az+B\xi]^*H_m[Az+B\xi]-z^*H_mz+F_m(C^*z,\xi)<0,\\ \forall\ z\in\mathbf{R}^n,\ \forall\ \xi\in\mathbf{R}^l,\ |z|+|\xi|\neq 0.\end{gathered} \tag{11}$$

Putting in (11) $\xi=0$ and taking into account (7), we get an inequality

$$\left(\frac{1}{\lambda_m}Az\right)^*H_m\left(\frac{1}{\lambda_m}Az\right)-z^*H_mz<0,\quad \forall\ z\in\mathbf{R}^n,\ z\neq 0.$$

From here and the condition 1) of the theorem by the lemma of Shepeljavyi [41] it follows that the matrix H_m has no less than m negative eigenvalues.

From relations (8) and (11) inequality (2) follows. Thus, here all conditions of Lemma 2 are satisfied. Theorem 1 is proved.

Now we shall suppose that instead of conditions (7) and (8) the following inequalities hold:

$$F_m(0,\xi) \geq 0, \quad \forall\, \xi \in \mathbf{R}^l, \tag{12}$$

$$F_m\big(U(k)^*\xi,\xi\big) \geq 0, \quad \forall\, \xi \in \mathbf{R}^l, \ \forall\, k \in \mathbf{Z}, \ k > 0. \tag{13}$$

Theorem 2. *Let* $\det A(k) \neq 0, \ \forall\, k \in \mathbf{Z}$ *and for some system of positive numbers* $\theta_1 \leq \ldots \leq \theta_n$ *the following conditions hold:*

1) a matrix $A\theta_m^{-1}$ *has* m *or more eigenvalues, situated inside of the circle of radius* 1 *with its centre at zero point,*

2) for all $p \in \mathbf{C}^1$ *such that* $|p| = 1$ *the following inequality*

$$F_m\big(\sigma, -W(\theta_m p)^*\sigma\big) < 0, \quad \forall\, \sigma \in \mathbf{C}^r, \ \sigma \neq 0 \tag{14}$$

is satisfied.

Then for singular values $\rho_{n-m}(k)$ *of the fundamental matrix* $X(k)$ *of system (1) there exists a number* γ_m *such that*

$$\rho_{n-m+1}(k) \leq \gamma_m \theta_m^k, \quad \forall\, k \in \mathbf{Z}, \ k > 0. \tag{15}$$

Proof. According to the conditions 2) of the theorem by the Kalman-Szego lemma [37] there exists a symmetric real matrix M_m, for which

$$\begin{gathered}\frac{1}{\theta_m^2}[A^*z + C\sigma]^* M_m [A^*z + C\sigma] - z^* M_m z + F_m(\sigma, B^*z) < 0, \\ \forall\, z \in \mathbf{R}^n, \ \ \forall\, \sigma \in \mathbf{R}^r, \ |z| + |\sigma| \neq 0.\end{gathered} \tag{16}$$

Putting in (16) $\sigma = 0$ and taking into account (12), we obtain an inequality

$$\left(\frac{1}{\theta_m}A^*z\right)^* M_m \left(\frac{1}{\theta_m}A^*z\right) - z^* M_m z < 0, \ \ \forall\, z \in \mathbf{R}^n, \ z \neq 0.$$

Hence, from the condition 1) of the theorem and from the fact that spectrums A and A^* are symmetric with respect to imaginary axis we obtain by the lemma of Shepeljavyi [41] that the matrix M_m has no less than m positive eigenvalues.

From relations (13) and (16) the inequality

$$\frac{1}{\theta_m^2}\big[A^*z + CU(k)^*B^*z\big]^* M_m \big[A^*z + CU(k)^*B^*z\big] - \\ - z^*Mz \le -F_m\big(U(k)^*B^*z, B^*z\big) \le 0, \quad \forall\, z \in \mathbf{R}^n$$

follows. The last inequality means that inequality (4) is satisfied. Thus, all conditions of Lemma 3 are satisfied and, consequently, estimate (5), which coincides with (15), can be found.

Now we shall suppose that $l = r = 1$ and for some numbers $\mu_1,\ \mu_2$ such that $\mu_1\mu_2 \le 0$ the inequality

$$\mu_1 \le U(k) \le \mu_2$$

is satisfied.

Theorem 3 (an analog of the circle criterion). *Let for some set of positive numbers $\lambda_1 \ge \ldots \ge \lambda_n$ condition 1) of Theorem 1 be satisfied and for all $p \in \mathbf{C}^1$ such that $|p| = 1$ the inequality*

$$Re\ \Big[\big(1 + \mu_1\overline{W(\lambda_m p)}\big)\big(1 + \mu_2 W(\lambda_m p)\big)\Big] > 0 \tag{17}$$

be satisfied. Then the statement of Theorem 1 is true.

For proving Theorem 3 it is sufficient to use Theorem 1 with the Hermitian form

$$F_m(\sigma,\xi) = (\mu_2\bar\sigma - \bar\xi)(\xi - \mu_1\sigma) + (\mu_2\sigma - \xi)(\bar\xi - \mu_1\bar\sigma). \tag{18}$$

Theorem 4 (an analog of the circle criterion). *Let the following inequalities be satisfied:*

$$\det A \ne 0, \quad 1 + W(0)\alpha \ne 0, \quad \forall\ \alpha \in [\mu_1, \mu_2]. \tag{19}$$

Suppose that for some set of positive numbers $\theta_1 \le \ldots \le \theta_n$ the condition 1) of Theorem 2 is satisfied and for all $p \in \mathbf{C}^1$ such that $|p| = 1$ an inequality

$$Re\ \Big[\big(1 + \mu_1\overline{W(\theta_m p)}\big)\big(1 + \mu_2 W(\theta_m p)\big)\Big] > 0 \tag{20}$$

holds. Then the statement of Theorem 2 is true.

Proof. It is easy to see that conditions (19) imply the nondegeneracy of a matrix $A(k) = A + BU(k)C^*$. For a Hermitian form

$$F_m(\sigma, \xi) = (\mu_2\bar{\xi} - \bar{\sigma})(\sigma - \mu_1\xi) + (\mu_2\xi - \sigma)(\bar{\sigma} - \mu_1\bar{\xi})$$

all conditions of Theorem 2 are satisfied. Really, we get

$$F_m(0, \xi) = -2\mu_1\mu_2\xi^2 \geq 0, \quad \forall\, \xi \in \mathbf{R}^1,$$

$$F_m\big(U(k)^*\xi, \xi\big) = 2\big(\mu_2 - U(k)\big)\big(U(k) - \mu_1\big)\xi^2 \geq 0, \quad \forall\, \xi \in \mathbf{R}^1,$$

$$\begin{aligned} &F_m(\sigma, -W(\theta_m p)^*\sigma) = \\ &= -2\mathrm{Re}\left[\big(1 + \mu_2\overline{W(\theta_m p)}\big)\big(1 + \mu_1 W(\theta_m p)\big)\right]|\sigma|^2 < 0, \\ &\forall\, \sigma \in \mathbf{C}^1, \qquad \sigma \neq 0. \end{aligned}$$

Therefore inequality (15) can be found.

Remark. It is evident that from Theorems 1-4 there follow the following estimates for the Lyapunov exponents ν_m:

$$\ln\theta_{n-m+1} \geq \nu_m \geq \ln\lambda_m.$$

Consider a system

$$x(k+1) = Ax(k) + Bf\big(C^*x(k)\big), \tag{21}$$

where A, B, C are respectively $n \times n$, $n \times l$ and $n \times r$ constant matrices and $f(\tau)$ is a continuously differentiable vector function.

Suppose that some compact set K is invariant with respect to the shift operator along the trajectories of system (21), i.e.

$$x(k, x_0) \in K, \quad \forall\, k \in \mathbf{Z}, \ \forall\, x_0 \in K.$$

Here $x(0, x_0) = x_0$.

Theorems 1-4 make it possible to obtain the upper estimates of the Hausdorff dimension $\dim K$ of the invariant compact set K. For this purpose we introduce the following notations:

$$\widetilde{K} = \big\{\tau \mid \tau = C^*x, \ x \in K\big\},$$

$$\mathcal{D}(\tau) = \left|\det\left(A + B\frac{\partial f(\tau)}{\partial\tau}C^*\right)\right|, \quad \mathcal{D} = \max_{\tau\in\tilde{K}}\mathcal{D}(\tau).$$

Here $\frac{\partial f(\tau)}{\partial\tau}$ is Jacobi matrix for the vector function $f(\tau)$ at a point τ.

Suppose, at first, that for Hermitian forms $F_m(\sigma,\xi)$ condition (7) is satisfied and condition (8) is replaced by the inequality

$$F_m\left(\sigma, \frac{\partial f(\tau)}{\partial\tau}\sigma\right) \geq 0, \quad \forall\ \sigma\in\mathbf{R}^r,\ \ \forall\ \tau\in\widetilde{K}. \tag{22}$$

Theorem 5. *Let for some set of positive numbers* $\lambda_1 \geq \ldots \geq \lambda_n$ *the conditions of Theorem 1 be satisfied. Suppose also that for some natural number* d *and for* $s\in[0,1]$ *an inequality*

$$\mathcal{D} < \lambda_{d+1}^{1-s}\lambda_{d+2}\cdots\lambda_n \tag{23}$$

holds. Then $\dim K < d+s$.

The estimate

$$\rho_1(k)\cdots\rho_{d+1}^s(k) \leq \beta\,\mathcal{D}^k\lambda_{d+1}^{k(s-1)}\lambda_{d+2}^{-k}\cdots\lambda_n^{-k}, \tag{24}$$

which follows from Theorem 1, is basis for the proof of this theorem. Here β is a number, $\rho_j(k)$ are singular numbers of fundamental matrices of systems obtained as a result of linearization of system (21) along trajectories $x(k,x_0)$, $x_0\in K$. Further, the proof of Theorem 5 repeats the scheme given in [23–25].

Let $l=r=1$ and for some numbers μ_1 and μ_2 such that $\mu_1\mu_2\leq 0$, an inequality

$$\mu_1 \leq \frac{df(\tau)}{d\tau} \leq \mu_2, \quad \forall\ \tau\in\widetilde{K} \tag{25}$$

be satisfied.

Theorem 6. *Let all conditions of Theorem 3 and inequalities* (23), (25) *be satisfied; then* $\dim K < d+s$.

This theorem is a corollary of Theorems 3 and 5.

Analogously, using Theorem 2 and 4, it is possible to formulate the following results.

Theorem 7. *Let* $\mathcal{D}(\tau)\neq 0$, $\forall\ \tau\in\widetilde{K}$, *inequality* (12) *be satisfied, and let*

$$F_m\left(\left(\frac{\partial f(\tau)}{\partial\tau}\right)^*\xi,\ \xi\right) \geq 0, \quad \forall\ \xi\in\mathbf{R}^l,\ \ \forall\ \tau\in\widetilde{K}. \tag{26}$$

Suppose also that for some set of positive numbers $\theta_1 \leq \ldots \leq \theta_n$ *all conditions of Theorem 2 are satisfied and for a natural number* d *and* $s \in [0,1]$ *the following inequality*

$$\theta_n \cdots \theta_{n-d+1} \theta_{n-d}^s < 1. \tag{27}$$

can be found. Then $\dim K < d + s$.

Suppose $l = r = 1$ and condition (25) hold.

Theorem 8. *Let all conditions of Theorem 4 and inequality* (27) *be satisfied. Then* $\dim K < d + s$.

Let us consider the system of Henon [19,44]

$$\begin{aligned} z(k+1) &= 1 - az(k)^2 + y(k), \\ y(k+1) &= bz(k), \end{aligned} \tag{28}$$

where a and b are positive numbers, $b < 1$.
In our notation,

$$A = \begin{pmatrix} 0 & 1 \\ b & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$f(\tau) = 1 - a\tau^2.$$

We shall suppose that for some invariant compact set K the following estimate is satisfied

$$K \subset \left\{ \begin{pmatrix} z \\ y \end{pmatrix} \,\middle|\, -\alpha \leq z \leq \alpha \right\}. \tag{29}$$

Here α is a positive number. In this case we may take

$$\mu_2 = -\mu_1 = 2a\alpha.$$

It is clear that here

$$W(p) = \frac{-p}{p^2 - b}, \quad \lambda_m^{-1} A = \begin{pmatrix} 0 & \lambda_m^{-1} \\ b\lambda_m^{-1} & 0 \end{pmatrix}.$$

It is easy to see that both eigenvalues of this matrix are situated outside of the unit circle if $\lambda_m < \sqrt{b}$.

In this case condition (17) takes the form

$$\left|W(\lambda_m p)\right|^2 < \mu_2^{-2}, \quad |p| = 1.$$

This inequality holds if

$$\frac{1}{\lambda_m \left|p^2 - \frac{b}{\lambda_m^2}\right|} < \mu_2^{-1}, \quad |p| = 1.$$

The last inequality is true if

$$\lambda_m^{-1} b - \lambda_m > \mu_2.$$

Thus, if

$$0 < \lambda_2 < -\frac{\mu_2}{2} + \sqrt{b + \frac{\mu_2^2}{4}},$$

all conditions of Theorem 3 are satisfied for $m = 2$.

In our case $\mathcal{D} = b, \quad d = 1, \quad m = 2.$ Then a number $s \in [0, 1]$, for which (23) holds, can be found from an inequality

$$s > 1 - \frac{\ln b}{\ln \lambda_2}.$$

From this estimate we obtain by Theorem 6 that

$$\dim K \le \max\left(1,\ 2 - \frac{\ln b}{\ln\left[\dfrac{-\mu_2}{2} + \sqrt{b + \dfrac{\mu_2^2}{4}}\right]}\right). \qquad (30)$$

Note that by a circle criterion

$$\left|W(p)\right|^2 < \mu_2^{-2}, \quad |p| = 1,$$

we obtain for system (28) the following result. If $\mu_2 < 1 - b$, then any solution $z(k),\ y(k)$, which is situated completely for all $k \ge 0$ in a band $\{-\alpha \le z \le \alpha\}$, tends to the state of equilibrium. Hence and from (30) we get finally the following estimate;

$$\dim K \le \begin{cases} 2 - \dfrac{\ln b}{\ln\left[\dfrac{-\mu_2}{2} + \sqrt{b + \dfrac{\mu_2^2}{4}}\right]} & \text{if } \mu_2 > 1 - b, \\[3ex] 0 & \text{if } \mu_2 < 1 - b. \end{cases}$$

Consider a case when $a = 1.4, \ \ b = 0.3$. It is shown in [19] that one can take $\alpha = 1.33$. Then from estimate (31) we obtain that $\dim K < 1.523$. This estimate have been obtain also for the Henon system in [20]. In [44] as a result of numerical experiment it was obtained that $\dim K = 1.24$.

CHAPTER III

ON STABILITY BY THE FIRST APPROXIMATION

What can be found in the neighbourhood of a solution of a nonlinear differential equation if the Lyapunov exponents of its linearization are known? Most of the classical results in this case are concerned with the investigation of the Lyapunov stability. A.M.Lyapunov has proved that if a linear system of the first approximation is a regular one and all of its Lyapunov exponents are negative, then the solution considered is asymptotically stable [29]. O.Perron has shown [36] that the requirement of regularity is essential. He gave an example of the Lyapunov unstable solution, a linearization along which was irregular and had negative Lyapunov exponents. N.G.Chetaev has obtained the same theorem for instability [9,10]: if a linear system of the first approximation is regular and at least one of its Lyapunov exponents is positive, then the solution is Lyapunov unstable.

When investigating the attractors of dynamical systems it is often necessary to study not individual solutions but certain ensambles of solutions [11,33,38,39,44]. By now many computational methods have been created and many computer experiments have been carried out, where the Lyapunov exponents of such ensambles have been computed [3,33,44]. As far as such analysis of attractors is concerned it is seems quite natural to replace the laborious requirement that the first approximation is regular by the requirement that the sign of Lyapunov exponents does not change at "small perturbations of initial data x_0". It turns out that such replacement is possible with the help of Theorems 1,2 which will be proved below.

Note that we have to use other, "not Lyapunov" notion of stability for the analysis of periodic solutions of autonomous systems. First of all, it is the notion of orbital stability introduced by A.Poincare (further, it will also be called the Poincare stability). In this case the well-known Andronov-Vitt theorem and its generalization given by B.P.Demidovich for the case of aperiodic trajectoriies [13] are concerned. B.P.Demidovich has shown that if a linear approximation along a certain bounded trajectory is regular, if it has one zero Lyapunov exponent, and all its other Lyapunov exponents are negative, then the trajectory considered is orbitally stable. Theorem 3, which will be proved below, permits us here also to take off the property of regularity, replacing it by the requirement that Lyapunov exponents remain negative for some naturally appearing linearization at small perturbations of the initial states x_0.

When studying instability of trajectories on attractors we have to use another concept of instability — Zhukovsky instability [24,26]. In order to illustrate the difficulties, appearing here, let us recall the main definitions of stability for a system

$$\frac{dx}{dt} = f(x), \qquad x \in \mathbf{R}^n, \quad f \in \mathbf{C}^2. \tag{1}$$

Definition 1. *A solution $x(t, x_0)$ of system (1) with initial data $x(0, x_0) = x_0$ is called Lyapunov stable if for any number $\varepsilon > 0$ there exists a number $\delta(\varepsilon) > 0$ such that for any vector y_0, satisfying the inequality $|x_0 - y_0| \le \delta(\varepsilon)$, and for any $t \ge 0$ the following inequality is satisfied: $\big|x(t, x_0) - x(t, y_0)\big| \le \varepsilon$. If furthermore, for some number δ_0 and all y_0 from the ball $\{y|\ |y - x_0| \le \delta_0\}$ the relation*

$$\lim_{t \to +\infty} \big|x(t, x_0) - x(t, y_0)\big| = 0$$

is satisfied, then the solution $x(t, x_0)$ is called Lyapunov asymptotically stable.

Here $|\cdot|$ is an Euclidean norm in $\mathbf{R}^n$.

Introduce now the following notation: $L^+(x_0)=\{x(t, x_0)|\ 0 \le t < +\infty\}$. Thus, the set $L^+(x_0)$ is a positive semitrajectory of system (1).

Definition 2. *A solution $x(t,x_0)$ of system (1) is called Poincare stable (or orbital stable), if for any number $\varepsilon > 0$ there exists a number $\delta(\varepsilon) > 0$ such that for any vector y_0 satisfying the inequality $|x_0 - y_0| \le \delta(\varepsilon)$ and for any $t \ge 0$ the inequality*

$$\rho\big(x(t, y_0),\ L^+(x_0)\big) \le \varepsilon \tag{2}$$

is satisfied. If furthermore, for some number δ_0 and all y_0 from the ball $\{y|\ |y - x_0| \le \delta_0\}$ the relation

$$\lim_{t \to +\infty} \rho\big(x(t, y_0),\ L^+(x_0)\big) = 0,$$

is satisfied, then the solution $x(t, x_0)$ is called Poincare asymptotically stable (or asymptotically orbitally stable).

Here $\rho(z, L)$ is the distance between a point z and a set L:

$$\rho(z, L) = \inf_{y \in L} |z - y|.$$

In order to define Zhukovsky stability, we shall need the following set of homeomorphisms

$$\mathrm{Hom} \ = \big\{\tau(\,\cdot\,)|\,\tau{:}[0,+\infty) \to [0,+\infty),\ \tau(0)=0\big\}.$$

Functions $\tau(t)$ from the set Hom will play a role of time reparametrizations for the trajectories of system (1).

Definition 3. *A solution $x(t,x_0)$ of system (1) is called Zhukovsky stable if for any number $\varepsilon > 0$ there exists a number $\delta(\varepsilon) > 0$ such that for any vector y_0, satisfying inequality $|x_0 - y_0| \le \delta(\varepsilon)$, a function $\tau(\,\cdot\,) \in \mathrm{Hom}$ can be found such that the inequality $\big|x(t,x_0) - x(\tau(t),y_0)\big| \le \varepsilon$, $\forall\, t \ge 0$ is satisfied.*

If furthermore, for some number $\delta_0 > 0$ and any y_0 from the ball $\{y|\ |x_0 - y_0| \le \delta_0\}$ a function $\tau(\,\cdot\,) \in \mathrm{Hom}$ can be found such that the relation

$$\lim_{t\to+\infty} \big|x(t,x_0) - x(\tau(t),y_0)\big| = 0$$

is satisfied, then the solution $x(t,x_0)$ is called Zhukovsky asymptotically stable.

In other words, Zhukovsky stability is Lyapunov stability under an adequate reparametrization each of the perturbed trajectories.

Recall that Lyapunov (or Poincare, Zhukovsky) instability means the negation of corresponding type of stability.

Obviously, Lyapunov stability implies Zhukovsky stability and Zhukovsky stability implies Poincare stability.

Note that in all these definitions we suppose as usual that all solutions considered are defined for all $t \in [0,+\infty)$.

Recall that all tree definitions are equivalent for equilibrium points. For periodic solutions it is easy to prove the equivalence of Poincare stability and Zhukovsky stability [24,26]. For nonlinear systems the situation often appears when a periodic solution is Zhukovsky asymptotically stable (and is consequently Poincare stable), but is Lyapunov unstable [24,26]. In this case we introduce a concept of asymptotic phase $c(y_0)$, which corresponds for Zhukovsky asymptotic stability to reparametrization $\tau(t) = t + c(y_0)$. Hence it follows that Lyapunov instability can not characterize a property of "repulsion of trajectories from each other" on such internally unstable objects as strange attractors.

Poincare instability also can not characterize the "repulsion of trajectories" on a strange attractor. Though the reason of it quite different. In numerical experiments often at least one semitrajectory $x(t,x_0)$ fills the attractor everywhere densely. (This is the situation with Lorenz attractor [11].) But then inequality (2) is satisfied for any $\varepsilon > 0$ and any point y_0 of the attractor. Consequently, all such trajectories $x(t,x_0)$ are Poincare stable.

But in experiments on a strange attractor in a course of time "repulsion of the trajectories from each other" is observed. Such "repulsion" corresponds to Zhukovsky instability. That is way Zhukovsky instability describs the behaviour of trajectories on strange attractors most adequately.

Various examples, demonstrating Zhukovsky unstable flows on two-dimensional compact manifolds, are given in [24,26].

A very important fact in connexion with Zhukovsky stability and instability is that certain linearizations, different from classical ones, appear here in a natural way. To describe these linearization we suppose that all considered semitrajectories $x(t, x_0)$, $t \geq 0$, of system (1) are disposed in a compact set G and $f(x) \neq 0$, $\forall\, x \in G$. In this case the following result on the existence of a special reparametrization of perturbed trajectories is true.

Lemma 1 [24,26]. *For any number $T > 0$ there exists a number $\delta(T) > 0$ such that for any vector y_0 from the set $\{y \mid |x_0 - y| \leq \delta(T),\ (y - x_0)^* f(x_0) = 0\}$ a differentiable function $\tau(\,\cdot\,) \in \mathrm{Hom}$ can be found, which satisfies the relation*

$$\big(x(t, x_0) - x(\tau(t), y_0)\big)^* f\big(x(t, x_0)\big) = 0, \quad \forall\ \ t \in [0, T]. \tag{3}$$

In addition, we have

$$\begin{aligned} \frac{d\tau(t)}{dt} = 1 - &\frac{f(x(t, x_0))^*}{|f(x(t, x_0))|^2} \left(\frac{\partial f}{\partial x}\,(x(t, x_0)) + \right. \\ &+ \left. \left(\frac{\partial f}{\partial x}\,\big(x(t, x_0)\big) \right)^* \right) \times \big(x(t, x_0) - x(\tau(t), y_0)\big) + \\ &+ O\big(|x(t, x_0) - x(\tau(t), y_0)|^2\big). \end{aligned} \tag{4}$$

Here $\frac{\partial f}{\partial x}\,(x(t, x_0))$ is the Jacobi matrix for the vector function f at the point $x(t, x_0)$. By $O(v)$ we denote a value, such that for sufficiently small v the estimate $|O(v)| \leq cv$ is satisfied, where c is a number. Here symbol $*$ denotes the conjugation.

From relation (4) it follows that for the reparametrization $\tau(t)$, indicated in the lemma, a linear system of the first approximation along the trajectory $x(t, x_0)$ has the form

$$\begin{aligned} \frac{dz}{dt} = \Bigg[&\frac{\partial f}{\partial x}\,(x(t, x_0)) - \frac{f(x(t, x_0))\, f(x(t, x_0))^*}{|f(x(t, x_0))|^2} \cdot \\ &\cdot \left(\frac{\partial f}{\partial x}\,(x(t, x_0)) + \left(\frac{\partial f}{\partial x}\,(x(t, x_0)) \right)^* \right) \Bigg] z, \end{aligned} \tag{5a}$$

$$f\big(x(t, x_0)\big)^* z = 0. \tag{5b}$$

Consider also a classic linearization

$$\frac{dy}{dt} = \frac{\partial f}{\partial x}\left(x(t,x_0)\right) y. \tag{6}$$

Lemma 2. *If $y(t)$ is a solution of system* (6), *then*

$$z(t) = \left(I - \frac{f(x(t,x_0))\, f(x(t,x_0))^*}{|f(x(t,x_0))|^2} \right) y(t)$$

is a solution of system (5a), (5b).

This statement can be verified by the substitution of the vector function $z(t)$ in the right and left sides of system (5).

Recall now that the definition of the Lyapunov exponents for a linear system is as follows

$$\frac{dx}{dt} = A(t)\, x, \quad x \in \mathbf{R}^n, \tag{7}$$

where $A(t)$ is a continuous $n \times n$-matrix. For this purpose denote by $X(t)$ a fundamental matrix of system (7) with initial data $X(0) = I$, where I is a unit matrix. Consider the square roots $\rho_i(t)$ of eigenvalues of a matrix $X(t)^*X(t)$, which are called singular numbers of a matrix $X(t)$. Futher, we shall suppose that $\rho_1(t) \geq \rho_2(t) \geq \cdots \geq \rho_n(t)$.

A very important and nearly obvious fact is that singular numbers of matrix $X(t)$ have a very simple geometrical interpretation. The operator $\mathbf{X}(t)$ transformes a ball of unit radius in the space $\mathbf{R}^n$ into an ellipsoid whose principal semiaxes coincide with singular numbers $\rho_i(t)$. Thus, singular numbers characterize the compressing and stretching properties of the operator $\mathbf{X}(t)$: $\mathbf{R}^n \to \mathbf{R}^n$. In particular, the estimate holds: $\rho_n(t)\,|x| \leq |X(t)x| \leq \rho_1(t)\,|x|, \ \forall\, x \in \mathbf{R}^n$.

The following simple statement resuls from the above.

Lemma 3. *For singular numbers $\alpha_i(t)$ of a matrix $X(t)S$ an estimate $\kappa_1\rho_i(t) \leq \alpha_i(t) \leq \kappa_2\rho_i(t)$ is true,* *where $\varkappa_1$ and $\varkappa_2$ are the minimal and the maximal singular numbers of the $n \times n$-matrix S respectively.*

Recall that a number

$$\nu_j = \ln \overline{\lim_{t \to +\infty}} \rho_j(t)^{1/t}. \tag{8}$$

is called a Lyapunov exponent ν_j of system (7)

From Lemma 3 it follows that in defining the Lyapunov exponents the condition $X(0) = I$ can be rejected since for the matrices $X(t)S$ and $X(t)$ under the condition $\det S \neq 0$ the value ν_j defined from formula (8) remains the same.

We begin now to investigate the Lyapunov stability and Zhukovsky stability for solutions $x(t, x_0)$ of system (1), assuming that $x_0 \in \Omega$, where Ω is a bounded open set in $\mathbf{R}^n$.

When investigating the Lyapunov stability let us consider here a more general case of a system

$$\frac{dx}{dt} = f(t, x), \quad x \in \mathbf{R}^n, \tag{9}$$

where $f(t, x)$ is a twice continuously differentiable vector function. Notice that without loss of generality, we can consider the solution $x(t, t_0, x_0)$ with the initial data $x(t_0, t_0, x_0) = x_0$ for $t_0 = 0$. Thus, we consider a solution $x(t, x_0)$ of system (9) with initial data $x(0, x_0) = x_0$. We shall assume here that for solutions of system (9) we have $x(t, x_0)$, $x_0 \in \Omega$, for singular numbers $\alpha_1(t, x_0) \geq \alpha_2(t, x_0) \geq \ldots \geq \alpha_n(t, x_0)$ of the fundamental matrix of the system

$$\frac{dy}{dt} = \left. \frac{\partial f(t, x)}{\partial x} \right|_{x = x(t, x_0)} y \tag{10}$$

and for a continuous function $\alpha(t)$ a relation

$$\alpha_1(t, x_0) \leq \alpha(t), \quad \forall\, t \geq 0, \ \forall x_0 \in \Omega \tag{11}$$

is satisfied.

Theorem 1. *Let the function $\alpha(t)$ be bounded on the interval $(0, +\infty)$. Then the solution $x(t, x_0)$, $x_0 \in \Omega$ is Lyapunov stable. If furthermore, $\lim\limits_{t \to +\infty} \alpha(t) = 0$, then the solution $x(t, x_0)$, $x_0 \in \Omega$, is Lyapunov asymptotically stable.*

Proof. Introduce here the following notations:

$$F_t y = x(t, y), \quad x(0, y) = y,$$

i.e., $F_t y$ is a shift operator along solutions of system (9),

$$(F_t y)' = X(t,y), \quad X(0,y) = I,$$

i.e., $(F_t y)'$ is the derivative with respect to y of the operator $F_t y$, which coincides with the fundamental matrix of the linear system (10).

It is well known [50] that according to the suppositions, made for any vectors y, z and a number $t \geq 0$, there exists a vector w, for which the following relations

$$|w - y| \leq |y - z|, \tag{12}$$

$$|F_t y - F_t z| \leq \left|(F_t w)'\right| |y - z| \tag{13}$$

are satisfied.

From here and from (11) it follows that for any y_0, for which $\{w|\ |w - x_0| \leq |x_0 - y_0|\} \subset \Omega$, the estimate

$$\begin{aligned} &|F_t x_0 - F_t y_0| \leq |x_0 - y_0| \sup \alpha_1(t,w) \leq \alpha(t)|x_0 - y_0|, \\ &\forall\, t \geq 0 \end{aligned} \tag{14}$$

is satisfied. Here the supremum is taken over all w from the ball $\{w|\ |w - x_0| \leq |x_0 - y_0|\}$.

The statement of Theorem 1 directly follows from estimate (14).

Suppose now that instead of inequality (11) for the fundamental matrix $X(t, x_0)$ of system (10) with the initial condition $X(0, x_0) = I$ and for some vector function $\xi(t)$ the following relations

$$\begin{aligned} &\left|\xi(t)\right| = 1, \qquad \forall\, t \geq 0, \\ &\max_i \inf_\Omega \left|X_i(t, x_0)\, \xi(t)\right| \geq \alpha(t), \quad \forall\, t \geq 0 \end{aligned} \tag{15}$$

are satisfied. Here $X_i(t, x_0)$ is the i-th row of the matrix $X(t, x_0)$.

Theorem 2. *Let for a function $\alpha(t)$ the relation*

$$\varlimsup_{t \to +\infty} \alpha(t) = +\infty \tag{16}$$

be satisfied. Then the solution $x(t, x_0)$, $x_0 \in \Omega$ is Lyapunov unstable.

Proof. Having fixed some pair $x_0 \in \Omega$ and $t > 0$, choose in any δ-neighbourhood of a point x_0 a vector y_0 such that

$$x_0 - y_0 = \delta\, \xi(t).$$

Assume that the number δ is so small that $\{w|\, |w - x_0| \leq \delta\} \subset \subset \Omega$.

Introduce also the following notation: $F_{it}z = x_i(t,z)$ is the i-th component of vector function $x(t,z)$.

It is well known [50] that for any fixed numbers t, i and vectors x_0, y_0 there exists a vector $w_i \in \mathbf{R}^n$, for which we have

$$\begin{aligned} &|x_0 - w_i| \leq |x_0 - y_0|, \\ &F_{it}x_0 - F_{it}y_0 = X_i(t,w_i)(x_0 - y_0). \end{aligned} \tag{17}$$

Using the last relation, we obtain the estimate

$$\begin{aligned} &|F_t x_0 - F_t y_0| = \sqrt{\sum_i \big|X_i(t,w_i)(x_0 - y_0)\big|^2} \geq \\ &\geq \delta \max\big\{\big|X_1(t,w_1)\xi(t)\big|, \ldots, \big|X_n(t,w_n)\xi(t)\big|\big\} \geq \\ &\geq \delta \max_i \inf \big|X_i(t,w)\,\xi(t)\big| \geq \delta\alpha(t). \end{aligned}$$

Here the infimum is taken over the set $w \in \big\{w \big|\ |x_0 - w| \leq \leq |x_0 - y_0|\big\}$.

From this estimate and from condition (16) it follows that for any positive numbers ε and δ there exists a vector y_0 and a number t, such that

$$|x_0 - y_0| \leq \delta, \quad |F_t x_0 - F_t y_0| \geq \varepsilon.$$

The last inequalities show that the solution $x(t,x_0)$ is Lyapunov unstable.

Let us discuss now the results obtained. Theorem 1 states the Lyapunov asymptotic stability for a flow of solutions with the initial data from Ω if their linearization have negative Lyapunov exponents. Here there are no effects discovered by O.Perron for irregular linearization of individual solutions [36]. Roughly speaking, the requirement of uniform exponential decrease "with respect to t_0" for singular numbers of fundamental matrix of the linearization (and this is one of the properties of regular stable linear systems) in classical theorems of Lyapunov and his successors [29,30] is replaced by the requirement of uniformity "with respect to x_0". Thus, the effects of Perron are possible only on the boundaries of the flow which is stable by the first approximation.

The requirements (15), (16) of Theorem 2 are in essence requirements of positiveness for at least one of the Lyapunov exponents of linearizations of the flow of solutions with the initial data from Ω with the following addition: "unstable directions $\xi(t)$" (or unstable manifolds) of these solutions coordinatewise continuously depend on the initial data x_0. Really, if this property take place, then considering, if it is necessary, the region Ω as a union of regions Ω_i of an arbitrary small diameter, where conditions (15) and (16) are satisfied, we obtain the Lyapunov instability for the flow of solutions under the initial data from Ω.

Consider now system (1) and linear approximation (5a), (5b) for solutions $x(t, x_0)$, $x_0 \in \Omega$ assuming that $x(t, x_0) \in \in \Omega$, $\forall\, t \geq 0$, and $f(x) \neq 0$, $\forall\, x \in \overline{\Omega}$. It is well known that $f(x(t, x_0))$ is a solution of system (6). Therefore it is possible to consider a linear independent system of solutions $f(x(t, x_0))$, $y_2(t), \dots, y_n(t)$ of system (6) and matrices

$$\begin{aligned} Y(t, x_0) &= \big(f(x(t, x_0)), \widetilde{Y}(t)\big), \\ \widetilde{Y}(t, x_0) &= \big(y_2(t), \dots, y_n(t)\big), \\ Z(t, x_0) &= \left(I - \frac{f(x(t, x_0))\, f(x(t, x_0))^*}{\big|f(x(t, x_0))\big|^2}\right) Y(t). \end{aligned}$$

Suppose that for singular numbers $\beta_1(t, x_0) \geq \ldots \geq \beta_n(t, x_0)$ of the fundamental matrix $Z(t, x_0)$ of system (5a), (5b) and for a continuous function $\beta(t)$ the estimate

$$\beta_1(t, x_0) \leq \beta(t), \qquad \forall\, t \geq 0, \quad \forall\, x_0 \in \Omega \tag{18}$$

is valid.

Theorem 3. *If*

$$\lim_{t \to +\infty} \beta(t) = 0,$$

then the solution $x(t, x_0)$, $x_0 \in \Omega$ is Zhukovsky asymptotically stable.

Proof. Fix a number T such that $\beta(t) \leq 1/2$, $\forall\, t \geq T$. By the number T by virtue of Lemma 1 we can choose a number $\delta(T) > 0$ such that for $x_0 \in \Omega$ and any y_0 satisfying the relations

$$|y_0 - x_0| \leq \delta(T), \quad (y_0 - x_0)^*\, f(x_0) = 0,$$

a differentiable function $\tau(t)$ can be found such that it satisfies relations (3), (4).

A number $\delta(T)$ can also be chosen such that the inclusion

$$\{y |\, |x_0 - y| \leq \delta(T)\} \subset \Omega$$

will be satisfied. Let us perform now the substitution of an independent variable $x(t) \to x\big(\tau(t)\big)$ in systems (1):

$$\frac{dx}{dt} = \frac{dx}{d\tau}\,\frac{d\tau}{dt} = f\big(x(\tau(t))\big)\,\frac{d\tau(t)}{dt} \tag{19}$$

It is clear that this system has some analog of the first integral (3), a linearization of system (19) along the solution $x(t, x_0)$ by virtue of (4) has form (5a), (5b). Therefore, repeating the reasoning used in proving Theorem 1, we obtain by an obvious equality $x(\tau(t), x_0) = x(t, x_0)$ the following

$$\big|x(T, x_0) - x\big(\tau(T), y_0\big)\big| \le \beta(T)\,|x_0 - y_0| \le \frac{1}{2}\,|x_0 - y_0|. \tag{20}$$

Considering further vector $q_0 = x(T, x_0)$ as new initial data and using again Lemma 1, we repeat all arguments given above on the interval $[0, T]$ for the solution $x(t, q_0)$. In this case, taking into account (20), we get:

$$\big|x(2T, x_0) - x\big(\tau(2T), y_0\big)\big| \le \frac{1}{4}|x_0 - y_0|.$$

Further, we shall continue the same reasoning and we obtain

$$\big|x(t, x_0) - x\big(\tau(t), y_0\big)\big| \le \sup_{t\ge 0} \beta(t)|x_0 - y_0|,$$
$$\lim_{t\to+\infty} \big|x(t, x_0) - x\big(\tau(t), y_0\big)\big| = 0.$$

These relations show that the solution $x(t, x_0)$ is Zhukovsky asymptotically stable.

Consider now a point $x_0 \in \Omega$ and a set $\Phi(x_0) = \big\{u \,|\, u \in \Omega,\ u^* f(x_0) = 0,\ |u - x_0| \le \gamma\big\}$. Here γ is a number.

Assume now that instead of inequality (18) for matrix $Z(t, u)$ previously introduced for some vector function $\xi(t)$, the relations

$$\begin{aligned} &\xi(t)^* f(x_0) = 0, \quad \big|\xi(t)\big| = 1, \quad \forall t \ge 0,\\ &\max_i \inf_{\Phi(x_0)} \big|Z_i(t, u)\,\xi(t)\big| \ge \alpha(t), \quad \forall\, t \ge 0 \end{aligned} \tag{21}$$

are valid.

Theorem 4. *Let for the function $\alpha(t)$ relation (16) be satisfied. Then the solution $x(t, x_0)$ is Zhukovsky unstable.*

Proof. Suppose the opposite, i.e., let the solution $x(t, x_0)$ be Zhukovsky stable. In this case there exists a number $\delta > 0$ such that for any y_0, satisfying inequality $|x_0 - y_0| \le \delta$, and $(x_0 - y_0)^* f(x_0) = 0$, there exists a reparametrization $\tau(t)$, for which relations (3) and (4) are satisfied for all $t \ge 0$ [24,26]. Thus, we can here on pass to system (19) for all $t \ge 0$. Repeating the reasoning of the proof of Theorem

2 and taking into account that system (5a), (5b) is a linearization of system (19) along the solution $x(t, x_0)$, we obtain that for any $t > 0$ there exists a vector y_0 such that $|x_0 - y_0| \le \delta$,

$$(x_0 - y_0)^* f(x_0) = 0,$$
$$\big|x(t, x_0) - x\big(\tau(t), y_0\big)\big| \ge \alpha(t)\,\delta.$$

From here, equality (3), and from condition (16) Zhukovsky instability of $x(t, x_0)$ the solution follows.

Note that Theorem 3 and 4 essentially differ from the results of [13,26] because they do not contain the requirement of regularity of the linearization. It is assumed instead that the stability (or instability) by the first approximation is preserved at small perturbations of the initial data x_0.

CHAPTER IV

FREQUENCY DOMAIN CRITERION OF EXISTENCE OF STRANGE ATTRACTORS FOR DISCRETE SYSTEMS

At the present time there are many ivestigations of chaotic behaviour of nonlinear discrete systems. Main results of these investigations may be found in books [33,35, 39,43,44,48,49]. The case of one-dimension dynamics is studied best of all in this case the phase space is either a segment of the straight line or a circle, or another set in $\mathbf{R}^1$ [6,33,39,43,44].

In the present chapter an approach connected with lower frequency estimates of highest Lyapunov exponents [21,22] is suggested. This approach is developed in the frame of the nonlinear control theory: main results are formulated in terms of transfer functions and frequency responses of linear parts of the systems. On the characteristics of nonlinear blocks $\varphi(\sigma)$ we impose conditions which are close to the traditional "inequalities of sector type" [17,24,25], typical for the theory of absolute stability of nonlinear control systems.

Howeover it can be found that for the existence of a strange attractor it is necessary "to turn inside out" such sector condition. Instead of customary and traditional conditions $\varphi(\sigma)/\sigma \in [k_1, k_2]$ or $\varphi'(\sigma) \in [k_1, k_2]$ for all $\sigma \in \mathbf{R}^1$ here we have to require that for almost all σ the condition $\varphi' \overline{\in} [k_1, k_2]$ is satisfied. In addition on certain sets of nonzero measure $\varphi'(\sigma) < k_1$ and $\varphi'(\sigma) > k_2$ both are satisfied. For continuous systems such conditions and corresponding frequency inequalities for transfer functions of the linear parts one select empty sets in the space of parameters. But for discrete systems such consideration permits us to obtain frequency domain criterions of existence of strange attractors. Let us consider examples of concrete discrete dynamic systems of the first and second order, for which frequency criterion, obtained here, defines conditions of existance of strange attractors.

For discrete systems we generally have no such obviousness of representations in the phase spaces, which is characteristic for the continuous systems with smooth trajectories filling this space. However, this deficiency can partly be corrected. For this purpose at the initial moment it is necessary to consider a segment of a straight line, connecting the initial data and its further iterations, i.e., the sequence of continuous curves The decrease or increase of the lengths of these curves gives the information about stability or instability or other properties of solutions discrete systems. The consideration of lengths of these continuous curves can also be used as a basis for the definition of stability or instability. This is especially important for noninjective mapping, defining the corresponding discrete systems (at present they are widely known as the generators of chaos). The matter is that in specific cases two trajectories may stick together: for the finite number of iterations. But the segment to be iterated, which connects the initial data, can have more and more increasing length even if this sticking is taken into account. Another interesting effect in studies of an exponential instability of strange attractors is the fact that in using the lower estimates of the lengths of curves to be iterated it is possible to do only with the definition of the notion of lower length of a curve (by analogy with the direction, which deals with the upper estimates of the Hausdorff dimension of attractors, where it is sufficient to introduce only the outer measure). This enables to study continuous curves without taking care about the extent of their smoothness. It turns out that in a number of cases the mechanism of stretching is strong enough and it can be revealed and evaluated by means of direct Lyapunov method, which can be connected with an efficient algebraic apparatus: the Kalman-Szego lemma and the Shepeljavyi lemma. The latter makes it possible to find the estimates of stretching of the lower lengths of curves, which are iterated, in terms of frequency domain inequalities, operating with transfer functions of linear blocks.

The effect of "sticking" is less strong than a 'stretching' in small neighbourhoods of the trajectories. This effect is strengthened with increasing the size of neighbourhood. Such an increase of the size of neighbourhood inevitably occurs when the effect of stretching is prevailing . At certain moment the neighbourhood becomes so much large that a 'sticking' in turn becomes strong that it completely compensates the stretching.

This situation makes it possible the coexistence of local exponential instability on the attractor and attracting properties (or at least of compactness) of the attractor itself. This situation makes it also possible to apply to the analysis of strange attractors of discrete systems the rich arsenal of methods, approaches and results of nonlinear analysis of the discrete control systems.

Consider a system

$$x(k+1) = f(x(k), k), \qquad x \in \mathbf{R}^n, \quad k \in \mathbf{Z}, \tag{1}$$

where $f(x,k)$ is a vector function: $\mathbf{R}^n \times \mathbf{Z} \to \mathbf{R}^n$, $\mathbf{R}^n$ is the Euclidean space of dimension n, $\mathbf{Z}$ is the set of integer numbers. We shall denote by $x(k, x_0)$ a solution of system (1) with the initial data $x(0, x_0) = x_0$. Denote by $S(r, x_0)$ a sphere of radius r with a centre at the point $x_0 \in \mathbf{R}^n$.

We shall need further a notion of lower length $l(\gamma)$ of a curve $\gamma(s)$, $s \in [0,1]$, where $\gamma(s)$ is a continuous function $[0,1] \to \mathbf{R}^n$.

Firstly, let us consider the case when $\gamma(s)$ is an injective mapping of the segment $[0,1]$ or a half-interval $(0,1]$ in $\mathbf{R}^n$. In other words, here

$\gamma(s_1) \neq \gamma(s_2)$ if $s_1 \neq s_2$. In this case, we define the lower length $l(s)$ of a curve γ in the following way.

Consider various finite and countable partitions of the segment $Q = [0,1]$ into sets of the form $Q_j = (\alpha_j, \beta_j)$, where

$$\begin{aligned} &Q_i \cap Q_j = \emptyset, \qquad \forall\, i \neq j, \\ &Q = \bigcup_j \overline{Q}_j. \end{aligned} \tag{2}$$

Here $\emptyset$ is an empty set. In this case we call the lower lenght $l(\gamma)$ of curve $\gamma(s)$ the following number (or $+\infty$)

$$l(\gamma) = \sup \sum_j \bigl|\gamma(\beta_j) - \gamma(\alpha_j)\bigr|,$$

where a supremum is taken over all possible partitions (2) of the segment $[0,1]$.

Suppose now that a set $[0,1]$ may be divided into a finite or a countable number of sets L_j $(j = 1, \dots, m)$ of the form (α_j, β_j), where

$$\begin{aligned} &L_i \cap L_j = \emptyset, \qquad \forall\, i \neq j, \\ &Q = \bigcup_j \overline{L}_j \end{aligned}$$

and a mapping $\gamma(\cdot)$ is injective on each set L_j.

In other words, here we consider the curves, being divided into continuous injective pieces $\gamma_j(s) : L_j \to \mathbf{R}^n$. We shall assume that the partition $\{L_j\}$ considered has one more property: for a subset $\{L_j\}^1$ of this partition the following condition holds:

$$\gamma(L_j) \cap \gamma(L_i) = \emptyset$$

for any $L_j \in \{L_j\}^1$ and for any $i \neq j$.

For subset $\{L_j\}^2$ the pieces $\gamma(L_j)$ stick with one another twice, i.e., for any $L_j \in \{L_j\}^2$ there exists the only $L_i \in \in \{L_j\}^2$, $i \neq j$ such that

$$\gamma(L_j) = \gamma(L_i)$$

and $\gamma(L_j) \cap \gamma(L_k) = \emptyset, \ \forall\, k \neq i, \ k \neq j$.

For a subset $\{L_j\}^3$ the corresponding pieces $\gamma(L_j)$ "stick with one another three times", i.e., for any $L_j \in \{L_j\}^3$ there exist precisely two different $L_i \in \{L_j\}^3, \ L_k \in \{L_j\}^3, \ i \neq j, \, k \neq j, \, i \neq k$ such that

$$\gamma(L_j) = \gamma(L_i) = \gamma(L_k)$$

and $\gamma(L_j) \cap \gamma(L_g) = \emptyset, \ \forall\, g \neq j, \ g \neq i, \ g \neq k$.

Continuing this process in the same way, define further a subset $\{L_j\}^4, \ldots, \{L_j\}^m$. Here m is some natural number for which

$$\{L_j\} = \bigcup_1^m \{L_j\}^k.$$

Assume that such m exists. It is clear that some of the subsets $\{L_j\}^k$ may be empty.

Definition 1. *The following number (or $+\infty$)*

$$l(\gamma) = \sum_{k=1}^m \frac{1}{k} \sum_{\{L_j\}^k} l(\gamma(L_j))$$

is called a lower length $l(\gamma)$ of the curve $\gamma(s)$.

The expression

$$\sum_{\{L_j\}^k} l(\gamma(L_j))$$

is the summation over all L_j, which belong to a subset $\{L_j\}^k$.

It is clear that for the case that $\gamma(\cdot)$ is a diffeomorphism, the lower length coincides with the usual length of a curve. The factors k^{-1} make it possible to take into account the case when some part of a curve "passes" with an increase of s" in two, three or more times. The necessity to exclude "the case of multiplicity" from consideration is typical for the study of chaos.

Note that there exists a simple estimate

$$l(\gamma) \geq \frac{1}{m}\, l_0(\gamma), \quad l_0(\gamma) = \sum_{\{L_j\}} l(\gamma(\overline{L}_j)),$$

that forms the basis of the proof of the criterion of instability, which will be formulated. Here $l_0(\gamma)$ may be interpreted as a lower length of a curve γ if we take into account "the multiplicity of sticking" as $l_0(\gamma)$ may be defined by formula (2) for any continuous curve $\gamma(s)$. In the case when it is difficult or it is impossible to take into consideration the value $l(\gamma)$ for noninjective $\gamma(s)$ we can introduce the value $l_m(\gamma) = l_0(\gamma)/m$ where m is an upper estimate of maximum number of pre-images of the points of a curve $\gamma(s)$. Further, let $l_m(\gamma)$ be called m-estimate of the lower length of the curve $\gamma(s)$.

Introduce now the following notation for a segment of the straight line in $\mathbf{R}^n$:

$$\Gamma(s, x_0, y_0) = s(y_0 - x_0) + x_0, \qquad s \in [0, 1].$$

Definition 2. *A solution $x(k, x_0)$ is called locally exponentially unstable if there exist positive numbers C, $\rho > 1$, ε such that for any $\delta \in (0, \varepsilon)$ a point $y_0 \in S(\delta, x_0)$ can be found, which satisfies the following conditions:*

1) there exist positive numbers $k \in \mathbf{Z}$ and $s \in [0, 1]$ such that

$$\big|x(k, x_0) - x(k, \Gamma(s, x_0, y_0))\big| > \varepsilon, \tag{3}$$

2) if for all $k \in [0, \mathcal{N}]$, $k \in \mathbf{Z}$, and $s \in [0, 1]$ the inequality

$$\big|x(k, x_0) - x(k, \Gamma(s, x_0, y_0))\big| \leq \varepsilon \tag{4}$$

is satisfied, then for these values of k the estimate

$$l\big(x(k, \Gamma(s, x_0, y_0))\big) \geq \rho^k \delta. \tag{5}$$

or

$$l_m(x(k, \Gamma(s, x_0, y_0))) \geq C\rho^k \delta \tag{6}$$

holds.

Note that Definition 2 for noninjective mappings $f(x,k)$ is some generalization of the usual definition of local exponential Lyapunov instability with Lyapunov exponent $\ln\rho$ (this definition is widely used for diffeomorphism). Note that in our case inequalities (5), (6) does not necessarily imply the property 1). For noninjective mappings, generally speaking, the following case is possible: $\delta \ll \varepsilon$, for all $k > 0$, the curves $x(k,\Gamma(s,x_0,y_0))$ remain in ε-neighbourhood of solution $x(k,x_0)$, and at the same time in this neighbourhood an exponential extension in directions, depending nonsmoothly on k takes place.

Consider system (1) with

$$f(x,k) = Ax + b\varphi(c^*x),$$

where A is a constant $n \times n$-matrix, b and c are constant n-vectors, $\varphi(\sigma)$ is a continuous piecewise differentiable function having ordinary descontinuities of derivative at the points $\sigma = \sigma_j$. We suppose that all the these points are isolated ones. In other words, we may find a number $\tau > 0$ such that

$$|\sigma_j - \sigma_i| \geq \tau, \quad \forall\, i \neq j.$$

We assume that the inequalities $\det(A + \varphi'(\sigma)bc^*) \neq 0,\ \forall\,\sigma \neq \sigma_j$, and

$$(\varphi'(\sigma) - \mu_2)(\mu_1 - \varphi'(\sigma)) \leq 0, \quad \forall\sigma \neq \sigma_j \tag{7}$$

hold. Here $\mu_1 < 0$, $\mu_2 > 0$ are some numbers.

Introduce a transfer function of system (1)

$$W(p) = c^*(A - pI)^{-1}b \qquad p \in \mathbf{C}^1$$

Theorem 1. *Suppose that for a number $\lambda > 2$ the following conditions hold:*

1) a matrix $\lambda^{-1}(A+\mu_1 bc^*)$ has one eigenvalue located outside of the circle of radius 1 with the centre at the zero point and $n-1$ eigenvalues lying inside of this circle,

2) for all values $p \in \mathbf{C}^1$ such that $|p| = 1$, the inequality

$$Re\left[(1+\mu_1\overline{W(\lambda p)})(1+\mu_2 W(\lambda p))\right] < 0 \tag{8}$$

is valid.

Then any solution of system (1) is locally exponentially unstable.

Proof. By the Kalman-Szego lemma [37] from the condition 2) of the theorem it follows that there exists a symmetric real matrix H such that

$$\begin{aligned}&\frac{1}{\lambda^2}[Az+b\xi]^*H[Az+b\xi]-z^*Hz-\\&-(\xi-\mu_2c^*z)(\mu_1c^*z-\xi)<0,\\&\forall\ z\in\mathbf{R}^n,\quad\forall\ \xi\in\mathbf{R}^1,\ \ |z|+|\xi|\neq0.\end{aligned}\tag{9}$$

Assuming in (9) $\xi=\mu_1c^*z$, we obtain the inequality

$$\begin{aligned}&\left(\frac{1}{\lambda}\left(A+\mu_1bc^*\right)z\right)^*H\left(\frac{1}{\lambda}\left(A+\mu_1bc^*\right)z\right)-z^*Hz<0,\\&\forall\ z\in\mathbf{R}^n,\quad z\neq0.\end{aligned}$$

Hence and from the condition 1) of the theorem by the lemma of Shepeljavyi [41] we obtain the matrix H which has one negative and $n-1$ positive eigenvalues.

Without loss of generality, suppose further that the vector z and the matrix H take the form

$$H=\begin{pmatrix}-1&0\\0&I\end{pmatrix},\qquad\begin{pmatrix}z_1\\z_2\end{pmatrix},$$

where I is a unit $(n-1)\times(n-1)$-matrix, $z_1\in\mathbf{R}^1$, $z_2\in\in\mathbf{R}^{n-1}$. In this case a quadratic form $V(z)=z^*Hz$ takes the form $V(z)=z^*Hz=|z_2|^2-z_1^2$.

We now take a number $\varepsilon<\tau/2$. From nondegeneracy of matrix $A+\varphi'(\sigma)bc^*$ in the bands $\{\sigma_j\leq\sigma<\sigma_{j+1}\}$ it follows that the preimage of any point of a mapping $Ax+b\varphi(c^*x)$, which is located in any ε-neighbourhood of any point of $\mathbf{R}^n$, is a set consisting either of one or two points.

Hence it follows that the estimation of number m in the definition of the lower lengths of image of the segment which can be found in such ε-neighbourhoods, followed by k-iterations is as follows: $m\leq2^k$.

Fix now a vector x_0 and choose for it y_0 such that $z_1(0)==\delta$, $z_2(0)=0$, where $z(0)=x_0-y_0$, $z_1(0)$ is the first component of the vector $z(0)$, $z_2(0)$ is $(n-1)$-vector of other components of the vector $z(0)$. Here δ is some fixed number such that $\delta<\varepsilon$. Thus, we have

$$\Gamma(s,x_0,y_0)=\begin{pmatrix}s\,z_1(0)\\0\end{pmatrix}+x_0,\qquad s\in[0,1].$$

Let us prove estimate (6), assuming that for $k\in[0,\mathcal{N}]$, $k\in\mathbf{Z}$, inequality (4) is true.

For this purpose we consider a function $\psi_k(s)= = c^* x(k, \Gamma(s, x_0, y_0)) - \sigma_{i(k)}$. In this case $\sigma_{i(k)}$ is a discontinuity of the derivative of $\varphi(\sigma)$ such that for some $s \in [0,1]$ the inequality $\psi_k(s) \geq 0$ is satisfied and for some other values s the inequality with the opposite sign is satisfied too.

Consider the following partition of the segment $[0,1]$ into mutual disjoint intervals of the form $(\alpha_{0j}, \beta_{0j})$. Here α_{0j} is a point for which $\psi_0(\alpha_{0j}) = 0$ and $\psi_0(s) > 0$ (or $\psi_0(s) < 0$) in some sufficiently small right-sided neighbourhood α_{0j} (i.e, $s > \alpha_{0j}$). For the point β_{0j} the relations $\psi_0(\beta_{0j}) = 0$ and $\psi_0(s) < 0$ (or $\psi_0(s) > 0$) are satisfied in some sufficiently small right-sided neighbourhood β_{0j} (i.e., $s > > \beta_{0j}$). In other words, α_{0j} and β_{0j} are the exit points of graphic $\psi_0(s)$ into the upper and the lower half-planes when increasing s. Besides, if it is necessary, we add the intervals $(0, \beta_{0j})$ and $(\alpha_{0j}, 1)$.

It is easy to see that the constructed system of intervals $M_{0j} = (\alpha_{0j}, \beta_{0j})$ has the following properties

$$M_{0j} \cap M_{0i} = \emptyset, \quad \bigcup_j \overline{M}_{0j} = Q.$$

In turn each a set M_{0j} can be partitioned in the same way for $\psi_1(s)$ into intervals $M_{0j,1j} = (\alpha_{0j,1j}, \beta_{0j,1j})$. Continuing this process further for $k=2, \ldots, N$ and reindexing $M_{0j,1j,\ldots}$, as a result, we obtain a partition of the segment $Q = [0,1]$ into the intervals M_j:

$$M_j \cap M_i = \emptyset, \quad \bigcup_j \overline{M}_j = Q.$$

It is important for us that on any $\overline{M}_j$ there exists a derivative$\varphi'(\sigma)$, where $\sigma = c^* x(k, \Gamma(s, x_0, y_0))$, $s \in \overline{M}_j$, $k = 0, \ldots, N$.

It is clear that under these conditions inequality (7) is also true. Therefore, from this fact and from inequality (9) it follows that for a difference $x(k, \Gamma(\beta_j, x_0, y_0)) - -x(k, \Gamma(\alpha_j, x_0, y_0))$, $k = 0, \ldots, \mathcal{N} - 1$, we can write an estimate

$$V\big(x(k+1, \Gamma(\beta_j, x_0, y_0)) - x(k+1, \Gamma(\alpha_j, x_0, y_0))\big) \leq$$
$$\leq \lambda^2 V\big(x(k, \Gamma(\beta_j, x_0, y_0)) - x(k, \Gamma(\alpha_j, x_0, y_0))\big).$$

Hence and from the special form of the matrix H it follows that

$$\big|x(k, \Gamma(\beta_j, x_0, y_0)) - x(k, \Gamma(\alpha_j, x_0, y_0))\big| \geq \lambda^k (\beta_j - \alpha_j),$$
$$\forall\, j, \qquad \forall\, k = 0, \ldots, \mathcal{N}.$$

Summing by j the left and right sides of this inequality and taking into account that it is possible to estimate $m \leq 2^k$, we obtain a relation

$$l_m\big(x(k,\Gamma(s,x_0,y_0))\big) \geq \frac{1}{2^k}\,\lambda^k\delta, \quad \forall k=0,\dots,\mathcal{N}. \tag{10}$$

Since $\lambda > 2$, estimate (6) is proved.

Inequality (3) may be proved in the same way, taking into account a special form of the matrix H. All the curves, considered here, may be projected on the line $\{z_2 = 0\}$. Since for the lower length of a projection an estimate being the same as (10) is satisfied, we can conclude that for some k and s, inequality (3) will be satisfied.

Let system (1) be represented in the form

$$\begin{aligned} &z(k+1) = Pz(k) + q\varphi(\sigma(k)),\\ &\sigma(k+1) - \sigma(k) = r^*z(k) + \rho\varphi(\sigma(k)), \end{aligned} \tag{11}$$

where P is a constant $(n-1)\times(n-1)$-matrix, r and q are constant n-vectors, $\varphi(\sigma)$ is a continuous function. It is clear that in this case the transfer function $W(p)$ for system (11) can be written in the form

$$W(p) = \frac{1}{p-1}\left[r^*(P-pI)^{-1}q - \rho\right].$$

It is well known that if $W(p)$ is a nondegenerated function (i.e., its denominator is a polynomial of degree n and does not have common zeros with its numerator [17], then both systems considered above are equivalent if $\det A = 0$.

The following results is well known and almost obvious.

Lemma. *If all eigenvalues of the matrix P are located inside of a unit circle with centre in the origin and $\varphi(\sigma)$ is a 2π-periodic function, then system* (11) *is Levinson dissipative in a cylindrical phase space* $\{z\in\mathbf{R}^n, \sigma \bmod 2\pi\}$.

Recall that the Levinson dissipativity in this case means that there exists a number R such that for any solution $z(k,z_0,\sigma_0), \sigma(k,z_0,\sigma_0)$ of system (11) the inequality

$$\overline{\lim_{k\to+\infty}}\,\big|z(k,z_0,\sigma_0)\big| \leq R \tag{12}$$

holds.

It is clear that in dissipativity region (12) there exists some global attractor (i.e., a compact invariant and globally attracting set). In case of a pointwise attraction it is, for example, a set of all ω-limit points of system (1).

Definition 3. *A global attractor K will be called a strange attractor, if for any $x_0 \in K$ a solution $x(k, x_0)$ is locally exponential unstable one.*

From the theorem and lemma the next statement follows.

Corollary. *Let all conditions of the theorem and the lemma be satisfied. Then there exists a strange attractor of system* (1).

Note that various conditions of Levinson dissipativity or the conditions of boundedness of solutions for discrete systems are given in [27]. Note also that various generalizations or weakenings of the conditions of the theorem are possible if of more precise estimates of sets L_j are used.

Example 1. Consider system (1) with $n = 1$, $A = 1$, $b = c = 1$, $\varphi(\sigma)$ being a 2π-periodic piecewise differentiable function satisfying conditions (7).

Here $W(p) = (1 - p)^{-1}$.

In this case the conditions 1) of the theorem take the form

$$|1 + \mu_1| > \lambda > 2, \tag{13}$$

and the conditions 2) are the following

$$\begin{aligned} &(\lambda \cos\omega)^2 + (1 - \lambda \sin\omega)^2 + \mu_1\mu_2 + \\ &+ (\mu_1 + \mu_2)(1 - \lambda \sin\omega) < 0, \quad \forall\omega \in \mathbf{R}^1. \end{aligned} \tag{14}$$

This inequality is equivalent to a condition

$$\lambda^2 + 1 + \mu_1\mu_2 + \mu_1 + \mu_2 + \lambda|2 + \mu_1 + \mu_2| < 0. \tag{15}$$

If we suppose that $\mu_2 = -\mu_1 = \mu$, then conditions (13) and (15) take a very simple form: $\mu > \lambda + 1 > 3$.

Thus, if $\mu > 3$, then there exists $\lambda > 2$ such that all conditions of the theorem are satisfied.

Thus, if $\mu > 3$, then in the phase space, which in this case is a circle, there is a strange attractor.

In Fig.1 one of the possible graphs of the function $\varphi(\sigma)$ is shown, for which the conditions of the theorem are satisfied. Here $\mu > 3$.

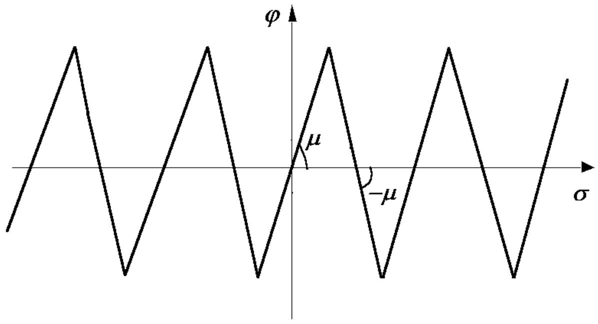

Fig. 1.

Example 2. Consider system (1) with $n = 1$, $A = 0$, $b = c = 1$, $\varphi(\sigma)$ which is a function piecewise differentiable satisfying conditions (7) with $\mu_2 = -\mu_1 = \mu$, and such that $\varphi(\sigma) \in \in [\alpha, \beta]$, $\forall \sigma \in [\alpha, \beta]$.

Here we may consider a compact phase space $[\alpha, \beta]$ and apply the theorem. It is easily seen that the condition 1) of the theorem holds if $\mu > \lambda > 2$. This condition is also sufficient for validity of theorem's condition 2).

Thus, for any function with graph, shown in Fig.1, where $\mu_2 = -\mu_1 = \mu > 2$ and which maps the segment $[\alpha, \beta]$ into itself, all conditions of the theorem are satisfied. Therefore, a local exponential instability takes place.

Note that a global stability of a fixed point for the system takes place for $\mu < 1$. Emphasize once more that some weakening of the conditions of the theorem may be achieved at the expense of more detailed information about sets L_j (see definition 1). For the case under consideration, it means that if iterations $k+1, \ldots, k+\mathcal{N}$, $\forall k \in \mathbf{Z}$ are known, for which all images of sufficiently small neighbourhoods of discontinuity points of the derivative σ_i do not intersect with a union of these neighbourhoods, then at the expense of this information we may improve the estimate $\mu > 2$. It is clear that the right side of such inequality will depend on $\mathcal{N}$.

Example 3. Consider system (11) with $n = 2$ and a transfer function

$$W(p) = \frac{\beta p}{(p-1)(p-\alpha)}, \qquad p \in \mathbf{C}^1,$$

where α and β are numbers, $\alpha \neq 0$, $|\alpha| < 1$, $\beta > > 0$, $\varphi(\sigma)$ is a 2π-periodic piecewise differentiable function satisfying condition (7) with $-\mu_1 = \mu_2 = \mu$.

The systems of this type describe some classes of rotators, being excited by the periodic shocks [44], and phase locked loop [27].

It is clear that here all conditions of the lemma are satisfied. Let us verify the conditions of the theorem.

Note that in this case the eigenvalues of a matrix $\lambda^{-1}(A + \mu_1 bc^*)$ are roots of a polynomial

$$(\lambda p - 1)(\lambda p - \alpha) - \mu\beta\lambda p.$$

It is easy to see that now the condition 1) of the theorem takes the form

$$\mu\beta + \alpha + 1 + \sqrt{(\mu\beta + \alpha + 1)^2 - 4\alpha} > 2\lambda,$$

and the condition 2) will be satisfied if

$$\mu\beta > \frac{(1 + \lambda)(\alpha + \lambda)}{\lambda}.$$

Thus, the system under consideration has a strange attractor in the cylindrical phase space if the following inequalities

$$(\mu\beta + \alpha + 1) + \sqrt{(\mu\beta + \alpha + 1)^2 - 4\alpha} > 4,$$
$$\mu\beta > \frac{3}{2}(\alpha + 2)$$

are satisfied.

Note that the frequency criterion formulated here can be extended in the natural way over the discrete systems with many nonlinearities.

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