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SOME PROPERTIES OF THE THERMOELASTIC PRESTRESSED MEDIUM GREEN FUNCTION

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Abstract. The dynamic coupled problem of excitation of harmonic oscillations on the layered prestressed thermoelastic body surface is considered. The body is under the action of the oscillating heat flow which is distributed on the surface in a certain region. It is a layer which is rigidly coupled to a half-space. The layer surface is assumed to be free from mechanical stress and outside the thermal stress region is thermal insulated. Thermal and mechanical effects induce initial stress state. The influence of different prestressing cases on the Green function poles distribution is investigated.

1. Formulation of the problem

We consider layered thermoelastic body, which is exposed to initial stress at uniform temperature. The initial stress is caused by mechanical and temperature impact. The body is a layer $-h \le |x_3| \le 0$ which is rigidly coupled to a half-space $|x_3| \le -h$. At the bound of layer and half-space the ideal heat contact condition is assumed. Medium vibration $\mathbf{u} = \{u_1, u_2, u_3, u_4\}$ ($\{u_1, u_2, u_3\}$ – vector of medium deformation, u_4 – temperature) is caused by either distributed on the surface in $\Omega = \{|x_1| \le 1, |x_2| \le \infty\}$ area tension field or heat flux $\mathbf{q}e^{-i\omega t}$ (ω - circular oscillation frequency, $\mathbf{q} = \{q_1, q_2, q_3, q_4\}$ is an extended vector of surface load, where $q_4 = -\lambda_3^{(1)} u_{4,3}^{(1)}$ is a heat flux). We assume that the surface out of loading region is stress-free and thermally insulated.

Stress-strain and temperature field relations and heat conduction equation [1-5]:

$$\nabla \cdot \mathbf{\Theta}^{(n)} = \rho^{(n)} \frac{\partial^2 \mathbf{u}^{(n)}}{\partial t^2}, \theta_{ij}^{(n)} = c_{ijkl}^{(n)*} u_{k,l}^{(n)} - \beta_{ij}^{(n)*} u_4^{(n)}, \tag{1}$$

$$\lambda_{ik}^{(n)} u_{4,ik}^{(n)} = \frac{\tau_1^{(n)}}{\tau_0} c_\varepsilon^{(n)} \rho^{(n)} \frac{\partial u_4^{(n)}}{\partial t} + \tau_1^{(n)} \beta_{ik}^{(n)*} \frac{\partial u_{k,i}^{(n)}}{\partial t}, \; n = 0,1. \tag{2}$$

Half-space parameters are designated as $n = 0$, layer parameters as $n = 1$.

Elastic and thermal parameters of materials in equations (1)-(2) with existing initial stress and heating are defined by next relations [3]:

$$c_{ijkl}^{(n)*}=\frac{\delta_{kj}}{2}c_{ilmm}^{(n)}\left(\nu_m^{(n)2}-1\right)+c_{ijkl}^{(n)}\nu_j^{(n)}\nu_k^{(n)}-\delta_{kj}\left(\tau_1^{(n)}-\tau_0\right)\beta_{il}^{(n)},\ \beta_{ij}^{(n)*}=\nu_j^{(n)}\beta_{ij}^{(n)}. \qquad (3)$$

Further we use the normalized parameters defined by next equations [1, 6]:

$$x_i'=\frac{x_i\cdot\omega *}{V_l},t'=\omega * t,V_P^{(n)}=\sqrt{c_{1111}^{(n)}/\rho^{(n)}},u_i'^{(n)}=\frac{u_i^{(n)}\cdot\rho^{(n)}\omega * V_P^{(n)}}{\beta_{11}^{(n)}\tau_0},i=1-3,\ \mathrm{E}^{(n)}=\frac{\tau_0\beta_{11}^{(n)2}}{\rho^{(n)}c_\varepsilon^{(n)}c_{1111}^{(0)}},$$

$$u_4'^{(n)}=\frac{u_4^{(n)}}{\tau_0},\ c_{ijkl}'^{(n)}=\frac{c_{ijkl}^{(n)}}{c_{1111}^{(0)}},\ \beta_{ij}'^{(n)}=\frac{\beta_{ij}^{(n)}}{\beta_{11}^{(0)}},\ \lambda_{ij}^{(n)}=\frac{\lambda_{ij}^{(n)}}{\lambda_{11}^{(0)}},\ \omega'=\frac{\omega}{\omega *},\ \omega *=\frac{c_\varepsilon^{(0)}c_{1111}^{(0)}}{\lambda_{11}^{(0)}}. \qquad (4)$$

In Eqs. (1) – (4) $c_{ijkl}^{(n)}$, $\lambda_{ij}^{(n)}$, $\alpha_{ij}^{(n)}$, $\beta_{kl}^{(n)}=\alpha_{ij}^{(n)}c_{ijkl}^{(n)}$ are the components of the elastic parameters tensor, thermal conductivity tensor, thermal expansion and thermoelasticity respectively, $\rho_0^{(n)}$ – is the material density in natural state, $c_\varepsilon^{(n)}$ – specific heat capacity at constant strain. τ_0 is the uniform temperature in natural state, $\tau_1^{(n)}$ – is the uniform temperature in a pre-stressed state, $\nu_k^{(n)}=1+\delta_k^{(n)}$, $\delta_k^{(n)}$ $(k=1,2,3)$ – are relative fiber extensions, $\mathrm{E}^{(n)}$ – thermoelastic relation constant, ω^* – normalized half-space frequency, $V_P^{(n)}$ – velocity of undeformed material longitudinal wave. The oscillations of the body are of a steady-state character so all the quantities are represented in the form: $f=f_0e^{-i\omega t}$. Asterisks and exponential factor have been suppressed for the convenience.

The special case when the oscillations are induced in the medium by the heat flux vertical component $\mathbf{q}_0=\{0\quad 0\quad 0\quad q_{40}\}$ distributed on the surface layer is considered. Then dimensionless boundary conditions are as follows:

$$x_3=0: \qquad \mathbf{q}^\tau=\begin{cases}\mathbf{q}_0^\tau\left(x_1,x_2\right),\ \left(x_1,x_2\right)\in\Omega,\\ 0,\ \left(x_1,x_2\right)\notin\Omega,\end{cases} \qquad (5)$$

$$x_3=-h: \qquad \begin{aligned}&\mathbf{u}^{(1)\tau}=\mathbf{u}^{(0)\tau},\\ &\mathbf{\Theta}^{(1)}\cdot\mathbf{n}=\mathbf{\Theta}^{(0)}\cdot\mathbf{n},\ -\lambda_{33}^{(1)}u_{4,3}^{(1)}=-\lambda_{33}^{(0)}u_{4,3}^{(0)}\end{aligned} \qquad (6)$$

$$x_3\to-\infty: \qquad \mathbf{u}^{(0)\tau}\to 0. \qquad (7)$$

2. Boundary problem Green's function

In order to study the effect of initial strain and preheating on Green's function poles the considered problem is assumed to be plane, i.e. all field quantities are independent of x_2:

$$u_2\equiv 0,\ f=f(x_1,x_3),\ \frac{\partial}{\partial x_2}f\equiv 0.$$

Taking a one-dimensional Fourier transform along x_1 axis to Eqs. (1)-(2) and (5)-(7) and it's solution will find in the form [2, 7]:

$$U_1^{(1)}(\alpha,x_3,\omega)=-i\alpha\sum_{k=1}^{3}f_{1k}^{(1)}\left(C_k\mathrm{sh}\sigma_k^{(1)}x_3+C_{k+3}\mathrm{ch}\sigma_k^{(1)}x_3\right),U_3^{(1)}(\alpha,x_3,\omega)=\sum_{k=1}^{3}f_{3k}^{(1)}\left(C_k\mathrm{ch}\sigma_k^{(1)}x_3+C_{k+3}\mathrm{sh}\sigma_k^{(1)}x_3\right),$$

$$U_4^{(1)}(\alpha,x_3,\omega)=\sum_{k=1}^{3}f_{4k}^{(1)}\left(C_k\mathrm{sh}\sigma_k^{(1)}x_3+C_{k+3}\mathrm{ch}\sigma_k^{(1)}x_3\right),\ -h\le x_3\le 0; \qquad (8)$$

$$U_1^{(0)}(\alpha,x_3,\omega)=-i\alpha\sum_{k=1}^{3} f_{1k}^{(0)}D_k e^{\sigma_k^{(0)}x_3},\ U_3^{(0)}(\alpha,x_3,\omega)=\sum_{k=1}^{3} f_{3k}^{(0)}D_k e^{\sigma_k^{(0)}x_3},$$
$$U_4^{(0)}(\alpha,x_3,\omega)=\sum_{k=1}^{3} f_{4k}^{(0)}D_k e^{\sigma_k^{(0)}x_3},\ \ x_3\le -h. \tag{9}$$

In Eqs. (8)-(9) σ_k are the roots computed numerically for each value of α and ω of the characteristic equation represented in [5,8]. Unknowns are obtained by substituting of (8)-(9) in the boundary conditions. Therefore, to find C_k, D_k we solve a system of linear algebraic equations:

$$\mathbf{LC}=\mathbf{Q},\ \mathbf{C}^{\tau}=\{C_1\quad C_2\quad C_3\quad C_4\quad C_5\quad C_6\quad D_1\quad D_2\quad D_3\}, \tag{10}$$

where $\mathbf{Q}=\{0\quad 0\quad Q_4\quad 0\quad 0\quad 0\quad 0\quad 0\quad 0\}$, Q_4 is the Fourier transform of q_{40}. The dispersion equation of the problem is: $\det\mathbf{L}=0$.

$$\mathbf{L}=\begin{pmatrix}
l_{11}^{(1)} & l_{12}^{(1)} & l_{13}^{(1)} & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & l_{31}^{(1)} & l_{32}^{(1)} & l_{33}^{(1)} & 0 & 0 & 0\\
l_{41}^{(1)} & l_{42}^{(1)} & l_{43}^{(1)} & 0 & 0 & 0 & 0 & 0 & 0\\
u_1^{(1)}s_1^{(1)} & u_1^{(1)}s_2^{(1)} & u_1^{(1)}s_3^{(1)} & u_1^{(1)}c_1^{(1)} & u_1^{(1)}c_2^{(1)} & u_1^{(1)}c_3^{(1)} & -u_1^{(0)}e_1^{(0)} & -u_1^{(0)}e_2^{(0)} & -u_1^{(0)}e_3^{(0)}\\
u_3^{(1)}c_1^{(1)} & u_3^{(1)}c_2^{(1)} & u_3^{(1)}c_3^{(1)} & u_3^{(1)}s_1^{(1)} & u_3^{(1)}s_2^{(1)} & u_3^{(1)}s_3^{(1)} & -u_3^{(0)}e_1^{(0)} & -u_3^{(0)}e_2^{(0)} & -u_3^{(0)}e_3^{(0)}\\
u_4^{(1)}s_1^{(1)} & u_4^{(1)}s_2^{(1)} & u_4^{(1)}s_3^{(1)} & u_4^{(1)}c_1^{(1)} & u_4^{(1)}c_2^{(1)} & u_4^{(1)}c_3^{(1)} & -u_4^{(0)}e_1^{(0)} & -u_4^{(0)}e_2^{(0)} & -u_4^{(0)}e_3^{(0)}\\
l_{11}^{(1)}c_1^{(1)} & l_{12}^{(1)}c_2^{(1)} & l_{13}^{(1)}c_3^{(1)} & l_{11}^{(1)}s_1^{(1)} & l_{12}^{(1)}s_2^{(1)} & l_{13}^{(1)}s_3^{(1)} & -l_{11}^{(0)}e_1^{(0)} & -l_{12}^{(0)}e_2^{(0)} & -l_{13}^{(0)}e_3^{(0)}\\
l_{31}^{(1)}s_1^{(1)} & l_{32}^{(1)}s_2^{(1)} & l_{33}^{(1)}s_3^{(1)} & l_{31}^{(1)}c_1^{(1)} & l_{32}^{(1)}c_2^{(1)} & l_{33}^{(1)}c_3^{(1)} & -l_{31}^{(0)}e_1^{(0)} & -l_{32}^{(0)}e_2^{(0)} & -l_{33}^{(0)}e_3^{(0)}\\
l_{41}^{(1)}c_1^{(1)} & l_{42}^{(1)}c_2^{(1)} & l_{43}^{(1)}c_3^{(1)} & l_{41}^{(1)}s_1^{(1)} & l_{42}^{(1)}s_2^{(1)} & l_{43}^{(1)}s_3^{(1)} & -l_{41}^{(0)}e_1^{(0)} & -l_{42}^{(0)}e_2^{(0)} & -l_{43}^{(0)}e_3^{(0)}
\end{pmatrix}. \tag{11}$$

In eqs. (11)

$$l_{1k}^{(n)}=-i\alpha(c_{3113}^{(n)}\sigma_k^{(n)}f_{1k}^{(n)}+c_{1313}^{(n)}f_{3k}^{(n)}),\ l_{3k}^{(n)}=-\alpha_1^{\ 2}c_{1133}^{(n)}f_{1k}^{(n)}+\sigma_k^{(n)}c_{3333}^{(n)}f_{3k}^{(n)}-\beta_3^{(n)}f_{4k}^{(n)},$$
$$l_{4k}^{(n)}=-\lambda_{33}^{(n)}\sigma_k^{(n)}f_{4k},\ u_p^{(n)}s_k^{(n)}=f_{pk}^{(n)}sh(-h\sigma_k^{(n)}),\ u_p^{(n)}c_k^{(n)}=f_{pk}^{(n)}ch(-h\sigma_k^{(n)}), \tag{12}$$
$$u_p^{(n)}e_k^{(n)}=f_{pk}^{(n)}e^{(-h\sigma_k^{(n)})},\ c_k^{(1)}=ch(-h\sigma_k^{(1)}),\ s_k^{(1)}=sh(-h\sigma_k^{(1)}),e_k^{(0)}=e^{(-h\sigma_k^{(0)})}.$$

After finding C_k, D_k from (10), the solution of the boundary value problem can be written as [2, 3]:

$$u_i^{(n)}(x_1,x_3)=\frac{1}{2\pi}\int_{-1}^{1}k_{i4}^{(n)}(x_1-\xi,x_3,\omega)q_4(\xi)\,d\xi,\qquad i=1,3,4, \tag{13}$$

$$k_{i4}^{(n)}(s,x_3,\omega)=\int_{\Gamma}K_{i4}^{(n)}(\alpha,x_3,\omega)e^{-i\alpha s}d\alpha, \tag{14}$$

where $K_{i4}^{(n)}(\alpha,x_3,\omega)$ are the matrix of Green's function elements, which are obtained from relations in [8].

3. Properties of the Green's function depending on the initial stress

The materials chosen for numerical calculations are described in [9]. The physical data for materials is given below:

$c_{1111}^{(1)}=27.52\cdot 10^{10}$ N/m², $c_{1133}^{(1)}=11.1\cdot 10^{10}$ N/m², $c_{3333}^{(1)}=27.52\cdot 10^{10}$ N/m², $c_{1331}^{(1)}=c_{3113}^{(1)}=$

$8.21\cdot 10^{10}$ N/m^2, $\beta_{11}^{(1)}=\beta_{33}^{(1)}=$ $4.6\cdot 10^{6}$ N/K/m^2, $c_{\varepsilon}^{(1)}$ = 462 J/kg/K, $\lambda_{11}^{(1)}=\lambda_{33}^{(1)}$ = 47.0 W/m/K, $\rho^{(1)}=$ 7823 kg/m^3, τ_0 = 280 K, $c_{1111}^{(0)}=$ $30.0\cdot 10^{10}$ N/m^2, $c_{1133}^{(0)}$ = $10.1\cdot 10^{10}$ N/m^2, $c_{3333}^{(0)}$ = $30.0\cdot 10^{10}$ N/m^2, $c_{1331}^{(0)}=c_{3113}^{(0)}$ = $15.75\cdot 10^{10}$ N/m^2, $\beta_{11}^{(0)}=\beta_{33}^{(0)}=$ $4.4\cdot 10^{6}$ N/K/m^2, $c_{\varepsilon}^{(0)}$ = 875 J/kg/K, $\lambda_{11}^{(0)}=\lambda_{33}^{(0)}$ = 58.0 0 W/m/K, $\rho^{(0)}=$ 3576 kg/m^3.

A feature of the thermoelastic body problems is the existence of a denumerable set of complex zeros and poles in the elements of function $K_{i4}^{(n)}(\alpha,0,\omega)$ [10]. Some of these possess a small imaginary part [8]. In order to construct solutions and effective investigation of the layered body dynamics requires a detailed study of the behavior of the poles, depending on the initial stress and pre-heating.

The calculated in a limited frequency range poles of the Green functions $K_{i4}^{(n)}(\alpha,0,\omega)$ with small imaginary part that play a decisive role in shaping the dynamic characteristics of the medium are shown on Fig. 1-2. On the axes are marked following dimensionless parameters: α is wavenumber, ω is normalized circular oscillation frequency. Fig. 1 shows the first five modes of the dispersion curves of layered thermoelastic half in the absence of initial deformations and heating. Fig. 2 shows the dispersion curves in the presence of the initial hydrostatic tension (a) and preheating (b). Poles for an unstrained environment marked by solid lines, for a medium with initial stresses - intermittent.

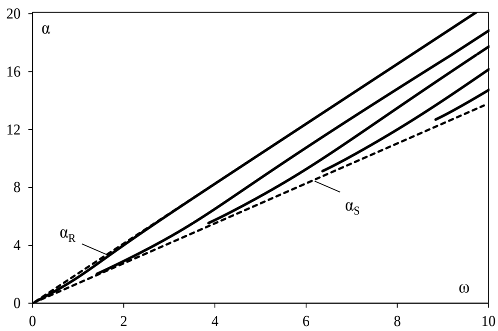
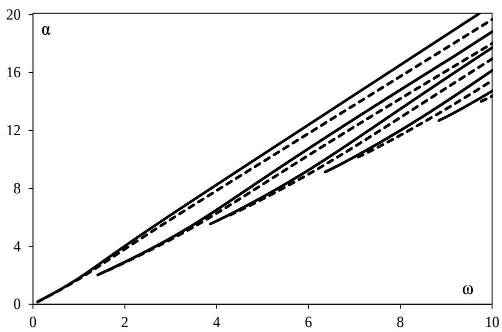
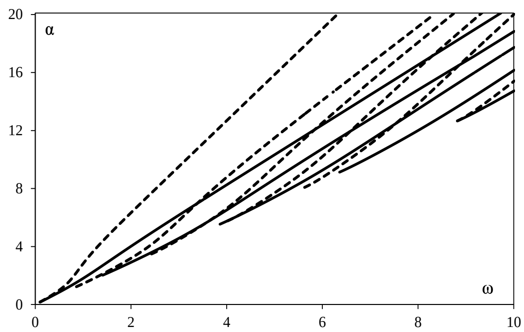


Fig. 1. Unstrained layered medium Green's function poles.



a)



b)

Fig. 2. Effect of hydrostatic initial strain and preheat to layered medium dispersion curves a) ν=1.01; b) *dt* =0.1.

Figure 1 shows that the poles with small imaginary part of the unstrained layered thermoelastic medium behave like poles of a layered elastic medium. Diagrams are stacked on the line between Rayleigh wave mode α_R in a homogeneous half-space with a layer material parameters and shear wave α_s produced in a homogeneous half-space.

Figure 2 shows that the initial stresses and heat significantly influence on the Green's function poles with small imaginary part distribution. Hydrostatic tension increases the phase velocity of waves in thermoelastic layered body, pre-heating reduces.

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