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INDENTATION OF AN AXISYMMETRIC PUNCH INTO AN ELASTIC TRANSVERSELY-ISOTROPIC HALF-SPACE WITH FUNCTIONALLY GRADED TRANSVERSELY-ISOTROPIC COATING

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Abstract. Axisymmetric frictionless contact problem of the theory of elasticity on indentation of a non-deformable punch into an elastic transversely-isotropic half-space with transversely-isotropic functionally graded coating is considered. Elastic moduli of the coating vary with depth according to arbitrary function. The technique based on integral transformations is used to reduce the problems to the integral equation. Special approximations for the kernel transform is used to obtain analytical solution of the integral equations. The solution is asymptotically exact for both large and small values of geometric parameter of the problem (relative layer thickness). A method of construction the compliance functions is presented for a case of arbitrary axisymmetric normal and tangential loadings.

1. Introduction

A large number of modern research focuses on contact problems for a functionally graded materials and coatings [1-3]. Often some special assumptions about the variation of elastic properties in the layer are accepted to make it possible obtaining analytical solutions of the corresponding differential equations, integral equations of the contact problems are usually solved numerically and the solutions are usually effective only for a special range of values of a geometrical parameter of the problem (relative layer thickness).

In the present paper, we consider an axisymmetric contact problem of the theory of elasticity on indentation of a non-deformable punch into an elastic transversely-isotropic half-space with transversely-isotropic functionally-graded coating. Elastic moduli of the coating vary with depth according to arbitrary functions. The main difference of contact problem considered in the paper and a similar problem for isotropic materials is in the scheme of construction of a kernel transform of integral equation (the compliance functions). Method of construction of compliance functions for a case arbitrary axisymmetric normal and tangential loading is presented in the paper. The solution of the indentation contact problem is constructed using the results obtained by the authors earlier for isotropic materials [4-8].

2. Construction the compliance functions

We consider an elastic inhomogeneous transversely isotropic half-space Ω. Cylindrical coordinate system r, φ, z with the z axis being the axis of symmetry for elastic moduli is chosen. Elastic moduli of the half-space vary with depth as:

$$c_{kj} = \begin{cases} c_{kj}^{(c)}(z) & -H \le z \le 0 \\ c_{kj}^{(s)} = const & -\infty < z < -H \end{cases}, \ (kj) = 11, 12, 13, 33, 44, \quad (1)$$

where $c_{kj}^{(c)}(z)$ are differentiable functions and $c_{kj}^{(s)}$ are constants. Hereafter, superscripts (c) and (s) correspond to the coating and to the substrate, respectively.

The coating and the substrate are assumed to be glued without sliding, so that the continuity conditions:

$$z = -H: \quad w^{(c)} = w^{(s)}, u^{(c)} = u^{(s)}, \sigma_z^{(c)} = \sigma_z^{(s)}, \tau_{rz}^{(c)} = \tau_{rz}^{(s)} \quad (2)$$

are satisfied, where u, w are the elastic displacements along the r and z axis respectively, $\sigma_r, \sigma_\varphi, \sigma_z, \tau_{rz}$ are the stresses.

We consider an axisymmetric distributed normal and tangential loading in the region $0 \le r \le a$ of the surface. Outside this region the surface is free of stress:

$$\sigma_z\big|_{z=0} = \begin{cases} -p(r), r \le a \\ 0, \quad r > a \end{cases}, \tau_{rz}\big|_{z=0} = \begin{cases} \tau(r), r \le a \\ 0, \quad r > a \end{cases}. \quad (3)$$

Linear constitutive equations for a transversely-isotropic material have the following form:

$$\sigma_r = c_{11}\frac{\partial u}{\partial r} + c_{12}\frac{u}{r} + c_{13}\frac{\partial w}{\partial z}, \sigma_\varphi = c_{12}\frac{\partial u}{\partial r} + c_{11}\frac{u}{r} + c_{13}\frac{\partial w}{\partial z},$$
$$\sigma_z = c_{13}\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) + c_{33}\frac{\partial w}{\partial z}, \quad \tau_{rz} = c_{44}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right). \quad (4)$$

The equilibrium equations due to the symmetry with respect to φ have the following form:

$$\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\varphi}{r} = 0, \quad \frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} = 0. \quad (5)$$

Use of the Hankel's transformations

$$u(r,z) = -\int_0^\infty \overline{u}(\gamma,z)\,\mathrm{J}_1(\gamma r)\gamma d\gamma, \ \{w(r,z), p(r), \tau(r)\} = \int_0^\infty \{\overline{w}(\gamma,z), \overline{p}(\gamma), \overline{\tau}(\gamma)\}\,\mathrm{J}_0(\gamma r)\gamma d\gamma \quad (6)$$

together with (4) reduces (5) to a system of two linear ordinary differential equations with variable coefficients:

$$\begin{cases} \dot{\overline{w}}(c_{13} + c_{44})\gamma + \overline{w}\dot{c}_{44}\gamma + \ddot{\overline{u}}c_{44} + \dot{\overline{u}}\dot{c}_{44} - \overline{u}c_{11}\gamma^2 = 0, \\ \ddot{\overline{w}}c_{33} + \dot{\overline{w}}\dot{c}_{33} - \overline{w}c_{44}\gamma^2 - \dot{\overline{u}}(c_{13} + c_{44})\gamma - \overline{u}\dot{c}_{13}\gamma = 0 \end{cases}. \quad (7)$$

Hereafter, $\dot{f}$ denotes differentiation of a function f with respect to z.

For the homogeneous substrate ($z \le -H$) system (7) is simplified to a following system of linear ordinary differential equations with constant coefficients, the general solution of this system is therefore:

$$\{\overline{u}^{(s)}, \overline{w}^{(s)}\}(\gamma, z) = D_1(\gamma)k_{\{11,31\}}e^{\alpha_1\gamma z} + D_2(\gamma)k_{\{12,32\}}e^{\alpha_2\gamma z}, \quad (8)$$

where $k_{ij} \in R, (i = 1,3, j = 1,2)$ are known constants, α_j are roots of the characteristic equation (generally they are different, the case of multiple root is principally similar to the following and was considered in [4] when the isotropic materials were modeled). $D_1(\gamma)$, $D_2(\gamma)$ are to be determined from the boundary conditions. The stresses and the displacements vanish at $z\to-\infty$ that is why constants corresponding to the negative roots are equal to zero.

Boundary conditions (2), (3) using (6) can be transformed to the following:

$$z=-H: \overline{w}^{(c)}=\overline{w}^{(s)}, \overline{u}^{(c)}=\overline{u}^{(s)}, \tag{9}$$

$$z=-H: c_{33}^{(c)}\dot{\overline{w}}^{(c)}-c_{13}^{(c)}\gamma\overline{u}^{(c)}=c_{33}^{(s)}\dot{\overline{w}}^{(s)}-c_{13}^{(s)}\gamma\overline{u}^{(s)}, \tag{10}$$

$$z=-H: c_{44}^{(c)}(\dot{\overline{u}}^{(c)}+\gamma\overline{w}^{(c)})=c_{44}^{(s)}(\dot{\overline{u}}^{(s)}+\gamma\overline{w}^{(s)}), \tag{11}$$

$$z=0: \quad c_{33}^{(c)}\dot{\overline{w}}^{(c)}-c_{13}^{(c)}\gamma\overline{u}^{(c)}=-\overline{p}(\gamma), c_{44}^{(c)}(\dot{\overline{u}}^{(c)}+\gamma\overline{w}^{(c)})=\overline{\tau}(\gamma). \tag{12}$$

Let us seek the $D_j(\gamma)$ in the form of linear combination of the applied loadings:

$$D_j(\gamma)=-M_{1j}(\gamma)\overline{p}(\gamma)+M_{3j}(\gamma)\overline{\tau}(\gamma), j=1,2. \tag{13}$$

Taking into account (8) let us introduce

$$\{\overline{u}_j^{(s)},\overline{w}_j^{(s)}\}(\gamma,z)=M_{j1}(\gamma)k_{\{11,31\}}e^{\alpha_1\gamma z}+M_{j2}(\gamma)k_{\{12,32\}}e^{\alpha_2\gamma z}. \tag{14}$$

We introduce the notations and rewrite the system (7) in the matrix form:

$$\vec{\mathbf{x}}=(x_1,x_2,x_3,x_4)^T, \ x_1=\overline{u}, \ x_2=\dot{\overline{u}}, \ x_3=\overline{w}, \ x_4=\dot{\overline{w}}, \ \dot{\vec{\mathbf{x}}}=\mathbf{A}\cdot\vec{\mathbf{x}}, -H\le z\le 0. \tag{15}$$

The superscript ‘T’ denotes the transposition of a matrix.
Let us seek the solution of (15) in the form: $\vec{\mathbf{x}}(\gamma,z)=-\overline{p}(\gamma)\cdot\vec{\mathbf{a}}_1(\gamma,z)+\overline{\tau}(\gamma)\cdot\vec{\mathbf{a}}_2(\gamma,z)$.

Using (10)-(14) vectors $\vec{\mathbf{a}}_j=\left(a_j^1,a_j^2,a_j^3,a_j^4\right)^T$, $j=1,2$ can be determined from a boundary value problem for a fixed γ as:

$$\dot{\vec{\mathbf{a}}}_j=\mathbf{A}\cdot\vec{\mathbf{a}}_j, j=1,2, \tag{16}$$

$$\begin{aligned} &z=-H: c_{33}^{(c)}a_j^4-c_{13}^{(c)}\gamma a_j^1=c_{33}^{(s)}\dot{\overline{w}}_j^{(s)}-c_{13}^{(s)}\gamma\overline{u}_j^{(s)},\\ &z=-H: c_{44}^{(c)}(a_j^2+\gamma a_j^3)=c_{44}^{(s)}(\dot{\overline{u}}_j^{(s)}+\gamma\overline{w}_j^{(s)}),\\ &z=0: \begin{matrix} c_{33}^{(c)}a_j^4-c_{13}^{(c)}\gamma a_j^1\\ c_{44}^{(c)}(a_j^2+\gamma a_j^3)\end{matrix}=\vec{e}_j, \vec{e}_1=\begin{pmatrix}1\\0\end{pmatrix}, \vec{e}_2=\begin{pmatrix}0\\1\end{pmatrix}.\end{aligned} \tag{17}$$

Variables $M_{kj}(\gamma),(k,j=1,2)$ can be determined from equations (9):

$$a_j^i(\gamma,-H)=M_{j1}(\gamma)k_{i1}e^{-\alpha_1\gamma H}+M_{j2}(\gamma)k_{i2}e^{-\alpha_2\gamma H}, i=1,3, j=1,2. \tag{18}$$

Introducing the notations: $L_{kj}(\gamma,z)=\gamma\, a_j^k(\gamma,z)/\Theta_{kj}$, $\Theta_{kj}=\lim_{\gamma\to\infty}\gamma a_j^k(\gamma,0)$

we finally obtain the displacements as a linear combination of Hankel’s transforms of the normal and tangential stresses acting on the surface:

$$\{\overline{u}^{(c)},\overline{w}^{(c)}\}(\gamma,z)=-\Theta_{\{11,31\}}\gamma^{-1}L_{\{11,31\}}(\gamma,z)\overline{p}(\gamma)+\Theta_{\{12,32\}}\gamma^{-1}L_{\{12,32\}}(\gamma,z)\overline{\tau}(\gamma), \tag{19}$$

where Θ_{kj} are certain constants depending on elastic properties at z=0.

Functions $L_{kj}(\gamma,z)$, k=1,3, j=1,2 are independent of the applied loading. Similar to A.K. Privarnikov [9] we will call them *compliance functions*. Generally, they can be calculated only numerically by solving the boundary value problem (16)-(18) for a fixed γ.

For z=0 the compliance functions are positive for every γ, $\lim\limits_{\gamma\to\infty} L_{kj}(\gamma,0)=1$ and $\lim\limits_{\gamma\to 0} L_{kj}(\gamma,0)$ depends solely on elastic moduli at z=0 and at z=-H (independent of variation of elastic properties inside the coating).

The presented scheme of construction the compliance functions can be used to reduce a mixed boundary value problems to an integral equation with functions $L_{kj}(\gamma,0)$ being kernel transforms.

3. Indentation problem

We consider a rigid punch with the flat circular base of radius a acting to the boundary of the half-space Ω in the region $z=0, r\le a$ (see fig. 1). The punch is subjected to the normal (centrally applied) force P. Friction between the punch and a layer is assumed to be absent. Under the action of force P the punch moves at a distance δ downward the z-axis. Outside of the punch, the surface is traction-free:

$$z=0:\ \tau_{zr}^{(c)}=0,\ \begin{cases}\sigma_z^{(c)}=0, & r>a,\\ w^{(c)}=-\delta, & r\le a.\end{cases} \tag{20}$$

The quantity of primary interest is the contact normal stresses under the punch: $\sigma_z\big|_{z=0}=-p_a(r),\ r\le a$.

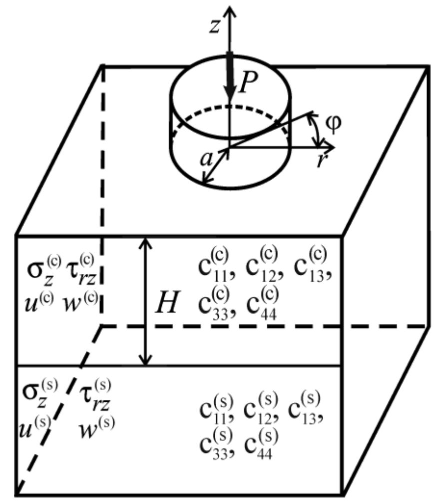

Fig. 1. Statement of the contact problem.

Using (4), (6), (20) an integral equation of the problem can be obtained [4, 10]:

$$\int_0^1 p(x)x\int_0^\infty L_{31}(u)\,\mathrm{J}_0(ur'\lambda^{-1})\,\mathrm{J}_0(ux\lambda^{-1})\,du\,dx=\lambda\Theta_{31}^{-1}\delta,\ r'\le 1, \tag{21}$$

where $\lambda=H/a$, $r'=r/\mathrm{a}$, $p(x)=p_a(xa)$.

By applying the bilateral asymptotic method [4, 10] an approximate solution of the integral equations (21) can be found in the analytical form:

$$p(r')=\frac{2}{\pi}\delta\Theta_{31}^{-1}\left\{\frac{1}{L_N(0)}\frac{1}{\sqrt{1-r'^2}}+\sum_{i=1}^{N}C_i\left(\frac{\mathrm{ch}(A_i\lambda^{-1})}{\sqrt{1-r'^2}}-\frac{A_i}{\lambda}\int_{r'}^{1}\frac{\mathrm{sh}(A_i\lambda^{-1}t)}{\sqrt{t^2-r'^2}}dt\right)\right\},r'<1. \tag{22}$$

Constants $\{C_i\}_{i=1}^N$ are found from the system of linear algebraic equations:

$$\sum_{i=1}^{N} C_i \lambda \frac{B_k \operatorname{ch}(A_i \lambda^{-1}) + A_i \operatorname{sh}(A_i \lambda^{-1})}{B_k{}^2 - A_i{}^2} + \frac{B_k{}^{-1} \lambda}{L_N(0)} = 0, \ k = 1,.., N,$$

$L_N(u)$ is the following approximation for the kernel transform of the integral equation:

$$L_N(u) = \prod_{i=1}^{N} \frac{u^2 + A_i^2}{u^2 + B_i^2} \approx L_{31}(u). \tag{23}$$

The solution (22) is asymptotically exact for both large and small values of relative layer thickness (i.e. $\lambda \to 0$ и $\lambda \to \infty$) [10]. The algorithm of constructing approximations of high accuracy and the analysis of the correspondence between the accuracy of the approximation for the kernel transform and the accuracy of the solution are described in [11].

4. Conclusion

For the spherical or conical punches, the compliance functions do not change. Solutions of the corresponding integral equations (similar to (21) with different right part) were constructed by the authors earlier [6, 7].

The compliance functions in the case of electroelastic, thermoelastic, magnetoelastic materials can be constructed using similar approach.

Bilateral asymptotic method used in the paper make it possible to construct an approximated solution of high accuracy for a wide range of mixed boundary value problems even for functionally-graded coatings of complicated structure 0, and in the case of sufficiently soft or hard coatings (elastic properties of the substrate and the coatings differ in few orders) [5].

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