Materials Physics and Mechanics 32 (2017) 353-369 Received: October 17, 2017

ON ONE CLASS OF APPLIED GRADIENT MODELS WITH SIMPLIFIED BOUNDARY PROBLEMS

S.A. Lurie[1,2*], P. A. Belov[1], Y.O. Solyaev[1], E.C. Aifantis[3,4,5,6,7]

[1]Institute of Applied Mechanics of Russian Academy of Science, Moscow, Russia

[2]Lomonosov Moscow State University Moscow, Russia

[3]Aristotle University of Thessaloniki, Thessaloniki 54124, Greece,

[4]Michigan Technological University, Houghton, MI 49931, United States,

[5]BUCEA, Beijing 100044, China,

[6]ITMO University, St. Petersburg 197101, Russia,

[7]Togliatti State University, Togliatti 445020, Russia

*e-mail: salurie@mail.ru

Abstract. We consider the generic gradient elasticity theory of Mindlin-Tupin and try to establish a class of applied models of gradient elasticity, for which the boundary value problems of the gradient theory with static boundary conditions are divided into a sequence of two subtasks, one of which is classical. Such applied models are very effective in applications, because their solutions reduce exactly to a consistent solution of boundary value problems of the second and not of the fourth order. We consider gradient theories with a general structure of tensors of gradient modules that satisfy potentiality conditions and additional symmetry conditions, which is considered as a criterion of correctness.

It is shown that their gradient tensors of the elastic modules are represented in the form of an expansion with respect to the tensor basis of five sixth-rank tensors, three of which satisfy a special property. Each of these basis tensors is represented as a convolution of fourth-rank tensors, and the corresponding quadratic form is a convolution of vectors.

It is shown that for the traditional gradient Mindlin-Tupin theory, the "classical" static conditions on the body surface are not satisfied locally. However, if the gradient modules are represented as a convolution of the "classical" tensors of elastic moduli, then the set of the boundary value problems of such gradient theory admits a full fractionation of the initial boundary value problem into two: the "classical" boundary value problem and the "cohesive" boundary value problem.

It is established the structure of the applied gradient models with such property of separating boundary value problems. They are particular cases of gradient elasticity theories with gradient modulus tensors, representable in the form of an expansion in three basis tensors of the sixth rank, satisfying the properties of the representation in the form of convolution via fourth-rank tensors.

We formulated "vector" gradient Mindlin-Tupin model that preserves the classical form of static boundary conditions. Such a model leads to a specific variant of the gradient theory with a single non-classical modulus, or one-parametrical model. It is shown that the obtained gradient model can be considered as some generalization of the well-known applied theory GradEla providing for it the separation of boundary value problems.

Keywords: gradient theories, scale parameters, separation of boundary value problems, "classical" displacement field, "cohesive" displacement field.

1. Introduction

In the gradient theory of elasticity the density of potential energy depends not only on first derivatives of the displacement vector, but also on the second derivatives of the displacement vector (first derivatives of deformation tensor in the framework of the Midlin's Form II models [1]). So, the statement of the gradient theory includes not only classical moduli of elasticity but also physical constants which dimension are different from the classical ones by the square of length. The gradient theory of elasticity was first formulated in [1, 2]. It was shown that in the general case for an isotropic medium, the model contains seven material constants – two classical Lame parameters and five additional modules.

The development of continuum media models accounting for various micro/nanostructures parameters beyond the theory of classical elasticity appears to be crucial for the description of short-range interactions, cohesion forces, and also for the modeling of other size-dependent effects in the framework of generalized elasticity and plasticity theories. Applied gradient model was developed initially by Aifantis [3]. Robust gradient models were developed for gradient elasticity by Aifantis and co-workers [4-6]. Later it was shown that, within the framework of the gradient theory of elasticity, it is possible to eliminate of the singularities of crack tips [6-8] and dislocations [9-11], correctly describe wave dispersion [12-13] and scale effects for the composite materials [14-22] and others. In this case, usually there are used simplified versions of the gradient theory of elasticity, which contain fewer additional parameters. The determination of additional physical constants requires the involvement of specific experimental approaches [23, 24] or methods of the molecular-dynamics modeling [21, 25-27]. Usually, there are used the applied models that, instead of five modulus [1, 2], contain three additional parameters [23,28] or two parameters [29] or a single additional scale parameter [4-6, 30]. A detailed classification of simplified models of the gradient theory of elasticity was considered in a recent paper [31, 32].

At the present time, gradient theories are actively developed and are increasingly used in various applied problems. However, fundamental questions of the construction of these theories are also discussed. In particular, there are discussed the physical meaning of additional high-order stresses [31,32], the problem of the correct formulation of models of gradient bars and plates [33-36], the problem of the correct formulation of the equilibrium equations and boundary conditions [32, 36-38], the problem of constructing models with allowance for the requirement of symmetry conditions [29] .

In this paper we discuss the problem of constructing a gradient theory of elasticity, in which static boundary conditions and equilibrium equations are written in terms of the same tensor of generalized stresses. In this paper, such stresses are suggested to call as “classical” stresses but not the “total” stresses introduced using terminology of E. Aifantis because the equilibrium equations are a divergence of these stresses, and the boundary conditions represent their convolution with the unit vector of normal to the surface of the body. The class of gradient models considered in the paper is the most attractive from a practical point of view, since for such models the solution of boundary value problems can often be simplified, sequentially solving the classical problem of elasticity theory and then solving the problem for an equation of Helmholtz type in which the right-hand side is the classical solution. Note that the known one-parameter gradient theories (so-called GradEla, SSGET etc. [29, 36, 37]) do not satisfy these requirements and their variational formulation leads to the appearance of natural boundary conditions in the complicated form [29,32,36]. In this paper we show the possibility of constructing a theory of GradEla type that satisfies these requirements, but with an asymmetric tensor of stresses.

2. «Vector» gradient model

Let us consider the Lagrangian L of the Mindlin-Tupin model:

$$L = A - \frac{1}{2}\iiint (C_{ijmn} R_{i,j} R_{m,n} + C_{ijkmnl} R_{i,jk} R_{m,nl}) dV. \quad (1)$$

Here $C_{ijmn} = \lambda\delta_{ij}\delta_{mn} + \mu(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm})$ is the tensor of classical modulus, C_{ijkmnl} is the tensor of gradient modulus, δ_{ij} is the Kronecker delta, A - is the work of the external given forces in the volume and on the surface of the body, R_i is the displacement vector.

We write down the conditions that determine the properties of the tensor of gradient elastic modules:

1. Existence of the density of potential energy:

$$C_{ijkmnl} = (C_{ijkmnl} + C_{mnlijk}) / 2. \quad (2)$$

2. The symmetry condition, determined by the requirement of continuity of displacements:

$$C_{ijkmnl} = (C_{ijkmnl} + C_{ikjmnl} + C_{ijkmln} + C_{ikjmln}) / 4. \quad (3)$$

As a result, taking into account conditions (2) and (3), we establish the general structure of the tensor C_{ijkmnl}:

$$\begin{aligned} C_{ijkmnl} &= \\ &= C_1 (\delta_{ij}\delta_{kl}\delta_{mn} + \delta_{ik}\delta_{jn}\delta_{ml} + \delta_{ij}\delta_{kn}\delta_{ml} + \delta_{mn}\delta_{lj}\delta_{ik}) + \\ &+ C_2 (\delta_{ij}\delta_{km}\delta_{nl} + \delta_{mn}\delta_{li}\delta_{jk} + \delta_{ik}\delta_{jm}\delta_{nl} + \delta_{ml}\delta_{ni}\delta_{jk}) + \\ &+ C_3 (\delta_{in}\delta_{jl}\delta_{km} + \delta_{mj}\delta_{nk}\delta_{li} + \delta_{in}\delta_{mj}\delta_{kl} + \delta_{il}\delta_{jn}\delta_{mk}) + \\ &+ C_4 \delta_{im} (\delta_{jn}\delta_{kl} + \delta_{jl}\delta_{nk}) + \\ &+ C_5 (\delta_{im}\delta_{jk}\delta_{nl}). \end{aligned} \quad (4)$$

Consequently, in the general form, the gradient elastic modules of the Mindlin-Tupin model depend on five parameters.

We note that sometimes the symmetry requirement for the first two indices is imposed. Then three additional relations are introduced for the parameters of the gradient tensor of the elasticity modulus (4):

$$\begin{aligned} C_{ijkmnl}Э_{ijr} &= \\ &= (C_1 - C_2)(\delta_{ml}Э_{knr} + \delta_{mn}Э_{klr}) + \\ &+ (C_2 - C_5)\delta_{nl}Э_{kmr} + \\ &+ (C_3 - C_4)(\delta_{kl}Э_{nmr} + \delta_{nk}Э_{lmr}) = 0, \end{aligned}$$

where $Э_{ijr}$ is the permutation symbol.

In this case, the gradient part of the energy density of the general model is two-parametrical. Let's call such a gradient model a completely symmetric gradient model.

We propose to introduce the definitions of basis tensors of sixth rank:

$$C_{ijkmnl} = C_1\delta^1_{ijkmnl} + C_2\delta^2_{ijkmnl} + C_3\delta^3_{ijkmnl} + C_4\delta^4_{ijkmnl} + C_5\delta^5_{ijkmnl}$$

$$\begin{cases} \delta^1_{ijkmnl} = (\delta_{ij}\delta_{kn}\delta_{ml} + \delta_{mn}\delta_{lj}\delta_{ik} + \delta_{ij}\delta_{kl}\delta_{mn} + \delta_{ik}\delta_{jn}\delta_{ml}) \\ \delta^2_{ijkmnl} = (\delta_{ij}\delta_{km}\delta_{nl} + \delta_{mn}\delta_{li}\delta_{jk} + \delta_{ik}\delta_{jm}\delta_{nl} + \delta_{ml}\delta_{ni}\delta_{jk}) \\ \delta^3_{ijkmnl} = (\delta_{in}\delta_{jl}\delta_{km} + \delta_{mj}\delta_{nk}\delta_{li} + \delta_{in}\delta_{mj}\delta_{kl} + \delta_{il}\delta_{jn}\delta_{mk})\ . \\ \delta^4_{ijkmnl} = \delta_{im}(\delta_{jn}\delta_{kl} + \delta_{jl}\delta_{nk}) \\ \delta^5_{ijkmnl} = (\delta_{im}\delta_{jk}\delta_{nl}) \end{cases} \quad (5)$$

Basis tensors δ^1_{ijkmnl}, δ^2_{ijkmnl} and δ^5_{ijkmnl} in (5) have the same structure: each term in them is the product of three Kronecker tensors, one of which has both indices belonging to the first triple of the indices of the sixth-rank tensor C_{ijkmnl}, the second one has indices belonging to different triples of indices of the sixth-rank tensor C_{ijkmnl}, and the third one has indices belonging to the second triple of indices of the sixth-rank tensor C_{ijkmnl}. The basis tensors δ^3_{ijkmnl} and δ^4_{ijkmnl} also have the same structure, but it differs from the previous one: all three Kronecker tensors in them have indices belonging to different triples of indices of the sixth-rank tensor C_{ijkmnl} (one index is from the first triple, another is from the second triple of the indices of the tensor C_{ijkmnl}). The density of the gradient potential energy, as a result, is divided into the sum of two fundamentally different terms. The first term is determined by the first group of basis tensors δ^1_{ijkmnl}, δ^2_{ijkmnl} and δ^5_{ijkmnl}, contains, respectively, the modules C_1, C_2, C_5 and determines the quadratic form, composed of the components of two vectors $\Delta R_i, R_{k,ki}$. The second term is determined by the second group of basis tensors δ^3_{ijkmnl} δ^4_{ijkmnl}, and contains, respectively, the modules C_3, C_4 and determines a quadratic form composed of the components of the tensor of the third, but not of the first rank.

It can be shown, for example, that the completely symmetric theory of gradient deformation, and the theory of Aero-Kuvshinsky, which is considered the theory of gradient rotations, contain two types of basis tensors: one of the first type, constructed as a linear combination of basis tensors δ^1_{ijkmnl}, δ^2_{ijkmnl} and δ^5_{ijkmnl}, second of the second type, constructed as a linear combination of basis tensors δ^3_{ijkmnl} and δ^4_{ijkmnl}.

Further, we will concentrate on the particular cases of gradient models, which contain only basic tensors of the first type. Preference is given to this particular case, because all three basis tensors δ^1_{ijkmnl}, δ^2_{ijkmnl} and δ^5_{ijkmnl} can be represented as convolutions with respect to one index of two tensors of the fourth rank.

Theorem: "All three basis tensors δ^1_{ijkmnl}, δ^2_{ijkmnl} and δ^5_{ijkmnl}, can be represented as convolutions with respect to one index of two tensors of the fourth rank"

Proof. In each term of the basis tensor δ^1_{ijkmnl}, δ^2_{ijkmnl} and δ^5_{ijkmnl} there is a factor containing indices from different triples of the sixth-rank tensor. We represent it as a convolution of two tensors of Kronecker, for example: $\delta_{im} = \delta_{ia}\delta_{ma}$. In a similar way, we will deal with each Kronecker tensor containing indices from different triples:

$$\begin{aligned}
\delta^{1}_{ijkmnl} &= (\delta_{ij}\delta_{kn}\delta_{ml} + \delta_{mn}\delta_{lj}\delta_{ik} + \delta_{ij}\delta_{kl}\delta_{mn} + \delta_{ik}\delta_{jn}\delta_{ml}) = \\
&= \delta_{ij}(\delta_{kn})\delta_{ml} + \delta_{mn}(\delta_{lj})\delta_{ik} + \delta_{ij}(\delta_{kl})\delta_{mn} + \delta_{ik}(\delta_{jn})\delta_{ml} = \\
&= \delta_{ij}(\delta_{ka}\delta_{na})\delta_{ml} + \delta_{mn}(\delta_{la}\delta_{ja})\delta_{ik} + \delta_{ij}(\delta_{ka}\delta_{la})\delta_{mn} + \delta_{ik}(\delta_{ja}\delta_{na})\delta_{ml} = \\
&= (\delta_{ij}\delta_{ka})(\delta_{ml}\delta_{na}) + (\delta_{ik}\delta_{ja})(\delta_{mn}\delta_{la}) + (\delta_{ij}\delta_{ka})(\delta_{mn}\delta_{la}) + (\delta_{ik}\delta_{ja})(\delta_{ml}\delta_{na}) = \\
&= (\delta_{ij}\delta_{ka})(\delta_{ml}\delta_{na} + \delta_{mn}\delta_{la}) + (\delta_{ik}\delta_{ja})(\delta_{mn}\delta_{la} + \delta_{ml}\delta_{na}) = \\
&= (\delta_{ij}\delta_{ka} + \delta_{ik}\delta_{ja})(\delta_{mn}\delta_{la} + \delta_{ml}\delta_{na}) \\
\delta^{2}_{ijkmnl} &= (\delta_{ij}\delta_{km}\delta_{nl} + \delta_{mn}\delta_{li}\delta_{jk} + \delta_{ik}\delta_{jm}\delta_{nl} + \delta_{ml}\delta_{ni}\delta_{jk}) = \\
&= \delta_{ij}(\delta_{km})\delta_{nl} + \delta_{mn}(\delta_{li})\delta_{jk} + \delta_{ik}(\delta_{jm})\delta_{nl} + \delta_{ml}(\delta_{ni})\delta_{jk} = \\
&= \delta_{ij}(\delta_{ka}\delta_{ma})\delta_{nl} + \delta_{mn}(\delta_{la}\delta_{ia})\delta_{jk} + \delta_{ik}(\delta_{ja}\delta_{ma})\delta_{nl} + \delta_{ml}(\delta_{na}\delta_{ia})\delta_{jk} = \\
&= (\delta_{ij}\delta_{ka})(\delta_{nl}\delta_{ma}) + (\delta_{jk}\delta_{ia})(\delta_{mn}\delta_{la}) + (\delta_{ik}\delta_{ja})(\delta_{nl}\delta_{ma}) + (\delta_{jk}\delta_{ia})(\delta_{ml}\delta_{na}) = \\
&= (\delta_{ij}\delta_{ka} + \delta_{ik}\delta_{ja})(\delta_{nl}\delta_{ma}) + (\delta_{jk}\delta_{ia})(\delta_{mn}\delta_{la} + \delta_{ml}\delta_{na}) \\
\delta^{5}_{ijkmnl} &= (\delta_{im}\delta_{jk}\delta_{nl}) = \\
&= (\delta_{jk}\delta_{ia})(\delta_{nl}\delta_{ma})
\end{aligned} \tag{6}$$

As a result, the gradient model, built on basic tensors (6), takes the following form:

$$\begin{aligned}
&C_{ijkmnl} = \\
&= C_1(\delta_{ij}\delta_{ka} + \delta_{ik}\delta_{ja})(\delta_{mn}\delta_{la} + \delta_{ml}\delta_{na}) + \\
&+ C_2[(\delta_{ij}\delta_{ka} + \delta_{ik}\delta_{ja})(\delta_{nl}\delta_{ma}) + (\delta_{mn}\delta_{la} + \delta_{ml}\delta_{na})(\delta_{jk}\delta_{ia})] + \\
&+ C_5(\delta_{jk}\delta_{ia})(\delta_{nl}\delta_{ma}).
\end{aligned} \tag{7}$$

In the basis (6), the doubled density of the potential energy of curvature of displacement has the form:

$$C_{ijkmnl}R_{i,jk}R_{m,nl} = 4C_1 R_{i,ia}R_{m,ma} + 4C_2 R_{i,ia}\Delta R_a + C_5 \Delta R_a \Delta R_a. \tag{8}$$

The quadratic form (8) can be established using equations (7). This form is canonical, and positive definite.

We note that in the expression for the gradient part of the potential energy density there are convolutions of the components of two vectors $R_{i,ia}$ and ΔR_a. Therefore, in what follows, we shall call this particular three-parameter model the "vector" gradient theory of elasticity.

For such a theory, it is easy to establish conditions for positive definiteness. Indeed, in accordance with the Sylvester criterion, for (8), we obtain the following system of inequalities:

$$\begin{cases} C_1 > 0 \\ C_1 C_5 - C_2 C_2 > 0 \end{cases}. \tag{9}$$

It follows from (9) that $C_5^T > 0$ too. Indeed, let us introduce instead of the modulus C_5, another modulus by the relation:

$$C_1 C_5 - C_2 C_2 = C^2. \tag{10}$$

As a consequence of (10), the second of the conditions (9) is identically satisfied. It also follows from (10):

$$C_1 C_5 = C^2 + C_2 C_2 > 0.$$

From the first condition of (9) and (10) we obtain:

$$C_5 = \frac{C^2 + C_2 C_2}{C_1} > 0.$$

3. On classical boundary conditions for the “vector” gradient model

Let us now consider in more detail the gradient theory, which is determined by the potential energy (1), (8) and which can be called the variant of the “vector” gradient theory. Using the relation (8), the density of the potential curvature energy in the "vector" theory (6) can be represented as a canonical positive definite quadratic form.

$$C_{ijkmnl}R_{i,jk}R_{m,nl} = 4C_1 R_{i,ia}R_{m,ma} + 4C_2 R_{i,ia}\Delta R_a + C_5 \Delta R_a \Delta R_a =$$
$$= 4\frac{C^2 + C_2 C_2}{C_5^T} R_{i,ia}R_{m,ma} + 4C_2 R_{i,ia}\Delta R_a + C_5 \Delta R_a \Delta R_a = \qquad (11)$$
$$= 4\frac{C^2}{C_5} R_{i,ia}R_{m,ma} + C_5 (\Delta R_a + \frac{2C_2}{C_5} R_{i,ia})(\Delta R_a + \frac{2C_2}{C_5} R_{i,ia}).$$

We can state that the “vector” theory in the general case contains three nonclassical moduli, under certain restrictions (9) due to positive definiteness of the canonical quadratic form of the density of the potential curvature energy (11).

We write the variational equation of the "vector" gradient model. From the requirement of stationarity of the Lagrangian (1) it follows that:

$$\delta L = \delta A - \iiint [C_{ijmn}R_{m,n}\delta R_{i,j} + C_{ijkmnl}R_{m,nl}\delta R_{i,jk}]dV =$$
$$= \delta A - \iiint [C_{ijmn}R_{m,n}\delta R_{i,j} + (4C_1 R_{m,ma} + 4C_2 \Delta R_a)\delta R_{i,ia} + (4C_2 R_{i,ia} + C_5 \Delta R_a)\delta\Delta R_a]dV =$$
$$= \delta A - \iiint [C_{ijmn}R_{m,n}\delta R_{i,j} +$$
$$+[4(C_1 + C_2)R_{m,ma} + (4C_2 + C_5)\Delta R_a]\delta R_{i,ia} +$$
$$+(4C_2 R_{i,ia} + C_5 \Delta R_a)\delta(\Delta R_a - R_{j,ja})]dV.$$

Using the relation:

$$(R_{a,jj} - R_{j,ja}) = (R_{m,jn}\delta_{ma}\delta_{nj} - R_{m,jn}\delta_{mj}\delta_{na}) = R_{m,jn}(\delta_{ma}\delta_{nj} - \delta_{mj}\delta_{na}) = R_{m,jn}Э_{mnk}Э_{ajk}\,,$$

we can found that the procedure of integrating by parts for the gradient part of the potential energy density will not require further transformations of the surface integral:

$$\delta L = \delta A - \iiint [C_{ijmn}R_{m,n}\delta R_{i,j} - (4C_1 + 8C_2 + C_5)\Delta R_{a,a}\delta R_{i,i} - C_5 \Delta R_{a,j}\delta R_{m,n}Э_{mnk}Э_{ajk}]dV -$$
$$-\oiint \{[4(C_1 + C_2)R_{m,ma} + (4C_2 + C_5)\Delta R_a]n_a\delta R_{i,i} +$$
$$+2(4C_2 R_{i,ia} + C_5 \Delta R_a)\delta(-R_{m,n}Э_{mnk}/2)n_j Э_{kja}\}dF.$$

Indeed, let’s introduce the classical definitions for the volume changing deformations $\theta = R_{i,i}$ and deformations of spins $\omega_k = -R_{m,n}Э_{mnk}/2$. These parameters determine on the surface independent variations of linear combinations of normal and tangential derivatives of displacements, which do not require further integrating by parts. As a result, the variational equation of the “vector” model takes the form:

$$\delta L = \iiint [(C_{ijmn}R_{m,n} - (4C_1 + 8C_2 + C_5)\Delta R_{a,a}\delta_{ij} - C_5 \Delta R_{a,c}Э_{ijk}Э_{ack})_{,j} + P_i^V]\delta R_i dV +$$
$$+\oiint \{[P_i^F - (C_{ijmn}R_{m,n} - (4C_1 + 8C_2 + C_5)\Delta R_{a,a}\delta_{ij} - C_5 \Delta R_{a,c}Э_{ijk}Э_{ack})n_j]\delta R_i - \qquad (12)$$
$$-[4(C_1 + C_2)R_{m,ma} + (4C_2 + C_5)\Delta R_a]n_a\delta\theta - 2(4C_2 R_{i,ia} + C_5 \Delta R_a)\delta(\omega_k n_j Э_{kja})\}dF = 0.$$

Equilibrium equations can be obtained from (12) as the Euler equations:

$$(C_{ijmn}R_{m,n} - (4C_1 + 8C_2 + C_5)\Delta R_{a,a}\delta_{ij} - C_5 \Delta R_{a,c}Э_{ijk}Э_{ack})_{,j} + P_i^V = 0. \qquad (13)$$

We call attention to the fact that the second-rank tensor, which divergence is equal to the external volume force in the equilibrium equations (13), can conditionally be called the “classical” stress tensor:

$$\tau_{ij} = C_{ijmn}R_{m,n} - (4C_1 + 8C_2 + C_5)\Delta\theta\delta_{ij} + 2C_5\Delta\omega_k Э_{ijk}. \tag{14}$$

Since this tensor (14) satisfies three classical equilibrium equations of elasticity theory:

$$\tau_{ij,j} + P_i^V = 0. \tag{15}$$

However, the “classical” stresses (14) differ from the “classical” ones, in the first place, in that this tensor is non–symmetric tensor, since the last term is antisymmetric when free indices are permuted. This term you can remove only if $C_5 = 0$. Therefore, the vector model can’t operate with the concept of true “classical” stresses.

On the other hand, the stress τ_{ij} satisfies not only the three classical equilibrium equations, but also the three classical static boundary conditions:

$$\oiint (P_i^F - \tau_{ij}n_j)\delta R_i dF = 0. \tag{16}$$

Indeed, boundary conditions for the considered variant of the “vector” theory break up into three pairs of alternative boundary conditions. The static boundary conditions (16) during variations of displacements completely coincide with the classical ones. Three pairs of alternative nonclassical boundary conditions break up into a pair of scalar alternative boundary conditions:

$$\oiint [4(C_1 + C_2)R_{m,ma} + (4C_2 + C_5)\Delta R_a]n_a\delta\theta dF = 0. \tag{17}$$

One of them is connected with variation of spherical tensor of deformation (see (17)). Two other pairs of alternative boundary conditions determine the possible work of some force vector $f_a = (4C_2R_{i,ia} + C_5\Delta R_a)$ on the variations of another (plane) vector $v_a = \omega_k n_j Э_{kja}$:

$$\oiint (4C_2R_{i,ia} + C_5\Delta R_a)\delta(\omega_k n_j Э_{kja})dF = 0. \tag{18}$$

It is not difficult to verify that the vector $v_a = \omega_k n_j Э_{kja}$ in (18) does not have a projection onto the normal to the surface, that is, lies in a tangent plane to the surface of the body $v_a n_a = (\omega_k n_j Э_{kja})n_a = \omega_k(n_j n_a Э_{jak}) \equiv 0$.

Let us return to the equilibrium equations and investigate the possibility of separating the equilibrium operator into a product of the classical equilibrium operator and an additional, nonclassical one. In other words, we will find out whether it is possible to represent the operator of equations (13) in the form:

$$[\mu\Delta(...)\delta_{ij} + (\mu+\lambda)(...)_{,ij}]\{(...)\delta_{jk} - l_\omega^2\Delta(...)\delta_{jk} - (l_\theta^2 - l_\omega^2)(...)_{,jk}\}R_k + P_i^V = 0. \tag{19}$$

By successively applying to the displacement vector R_k, first the operator in curly brackets (19), and then the operator in square brackets, we get:

$$\mu(\Delta R_i - R_{j,ji}) + (2\mu+\lambda)R_{j,ij} - \mu l_\omega^2\Delta(\Delta R_i - R_{j,ji}) - (2\mu+\lambda)l_\theta^2\Delta R_{j,ji} + P_i^V = 0 \tag{20}$$

Comparing (20) and (13), we find that the equations coincide if the parameters l_θ^2, l_ω^2 are related to nonclassical modules by the following relations:

$$\begin{cases} C_5 = \mu l_\omega^2 \\ 4C_1 + 8C_2 = (2\mu+\lambda)l_\theta^2 - \mu l_\omega^2 \end{cases}. \tag{21}$$

Applying the operator in curly brackets of equation (19) to the vector R_k (19), we obtain the definition of “classical” displacements U_i:

$$\begin{aligned} U_j &= \{(...)\delta_{jk} - l_\omega^2\Delta(...)\delta_{jk} - (l_\theta^2 - l_\omega^2)(...)_{,jk}\}R_k = \\ &= R_j - l_\omega^2(\Delta R_j - R_{k,kj}) - l_\theta^2 R_{k,kj}. \end{aligned} \tag{22}$$

Taking into account the definition (22), the equilibrium equations (19) take the form of the Lamé equations of the classical theory of elasticity in displacements:

$$[\mu\Delta(...)\delta_{ij}+(\mu+\lambda)(...)_{,ij}]U_j+P_i^V=0. \tag{23}$$

Since the linear differential operators in (19) are commutative, the equilibrium equations can be rewritten in the following equivalent form:

$$\frac{\mu}{l^2}\{(...)\delta_{ij}-l_\omega^2\Delta(...)\delta_{ij}-(l_\theta^2-l_\omega^2)(...)_{,ij}\}l^2[\Delta(...)\delta_{jk}+\frac{(\mu+\lambda)}{\mu}(...)_{,jk}]R_k+P_i^V=0. \tag{24}$$

The first of the operators in (24) is a generalized Helmholtz operator. Therefore, we can introduce a vector of "cohesive" displacements [14-16, 19, 20, 21], u_j:

$$\begin{aligned}u_j&=-l^2[\Delta(...)\delta_{jk}+\frac{(\mu+\lambda)}{\mu}(...)_{,jk}]R_k=\\&=-l^2[(\Delta R_j-R_{k,kj})+\frac{(2\mu+\lambda)}{\mu}R_{k,kj}].\end{aligned} \tag{25}$$

Taking into account the definition of u_j, (25) the equilibrium equations give the equilibrium equations of the “cohesive” field:

$$l_\omega^2(\Delta u_i-u_{j,ji})+l_\theta^2 u_{j,ji}-u_i+l^2\frac{P_i^V}{\mu}=0. \tag{26}$$

Let us consider the definitions (22) and (25) as a linear algebraic system with respect to the vortex field $(\Delta R_j-R_{k,kj})$ and the potential field $R_{k,kj}$:

$$\begin{cases}l_\omega^2(\Delta R_j-R_{k,kj})+l_\theta^2 R_{k,kj}=R_j-U_j\\ l^2(\Delta R_j-R_{k,kj})+\dfrac{(2\mu+\lambda)}{\mu}l^2R_{k,kj}=-u_j\end{cases} \tag{27}$$

It is easy to see that the equation system (27) can be rewritten in the following form:

$$\begin{cases}(\Delta R_j-R_{k,kj})=\dfrac{-\dfrac{(2\mu+\lambda)}{\mu}\dfrac{1}{l_\omega^2}(R_j-U_j)-\dfrac{l_\theta^2}{l_\omega^2}\dfrac{1}{l^2}u_j}{[\dfrac{l_\theta^2}{l_\omega^2}-\dfrac{(2\mu+\lambda)}{\mu}]}\\ R_{k,kj}=\dfrac{\dfrac{1}{l_\omega^2}(R_j-U_j)+\dfrac{1}{l^2}u_j}{[\dfrac{l_\theta^2}{l_\omega^2}-\dfrac{(2\mu+\lambda)}{\mu}]}\end{cases}. \tag{28}$$

The first of equations (28) determines the vortex field $(\Delta R_j-R_{k,kj})$. Its divergence is, by definition, equal to zero. Therefore, taking into account (22), (28) we can write:

$$R_{k,k}=U_{k,k}-\frac{\mu}{(2\mu+\lambda)}\frac{l_\theta^2}{l^2}u_{k,k}. \tag{29}$$

The second of the equations (28) determines the potential field $R_{k,kj}$. Its rotor is zero, by definition:

$$R_{m,n}Э_{mnr}=U_{m,n}Э_{mnr}-\frac{l_\omega^2}{l^2}u_{m,n}Э_{mnr}. \tag{30}$$

Accordingly, we can write the following equation for the rotor of the rotor:

$$(\Delta R_k-R_{m,mk})=(\Delta U_k-U_{m,mk})-\frac{l_\omega^2}{l^2}(\Delta u_k-u_{m,mk}). \tag{31}$$

Substituting (30) and (31) into (23), we obtain the general solution of the “vector” gradient theory R_i through two fundamental vectors, one of which is a vector of “classical” displacements U_i, and the second one is a vector of “cohesive” displacements u_i:

$$R_i = U_i + l_\omega^2(\Delta U_i - U_{k,ki}) + l_\theta^2 U_{k,ki} - \frac{l_\omega^2 l_\omega^2}{l^2}(\Delta u_i - u_{k,ki}) - \frac{\mu}{(2\mu+\lambda)}\frac{l_\theta^2 l_\theta^2}{l^2} u_{k,ki}. \quad (32)$$

Let us write down the tensor of stresses in displacements. Substituting vector of displacements R_i, with the help of equation (32), into (14) and taking into account the definitions (22), we can get:

$$\tau_{ij} = C_{ijmn}U_{m,n} +$$
$$+2\mu l_\theta^2[\frac{\mu}{(2\mu+\lambda)}\frac{1}{l^2}u_{k,k}]\delta_{ij} + \mu(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm})[l_\omega^2(\Delta U_{m,n} - U_{k,kmn}) + \quad (33)$$
$$+l_\theta^2 U_{k,kmn} - \frac{l_\omega^2 l_\omega^2}{l^2}(\Delta u_{m,n} - u_{k,kmn}) - \frac{\mu l_\theta^2 l_\theta^2}{(2\mu+\lambda)l^2}u_{k,kmn}] + 2\mu l_\omega^2 \Delta\omega_k Э_{ijk}.$$

Let’s make the following remark. In expressions (25), (26), we introduce a scale normalizing parameter when it’s determined the “cohesive” field vector u_i. We can assume without loss of generality that $l = l_\omega^2$. In the general case, both the system of equilibrium equations (24), the general solution of these equations and the expression for the stresses (33), are written in terms of “classical” displacements and “cohesive” displacements are determined only through two scale parameters l_ω^2 и l_θ^2. The spherical tensor of deformations $R_{k,k}$ and pseudo-vector of rotations $R_{m,n}Э_{mnr}$ (see equations (29) and (30)) are also written explicitly through “classical” displacements and “cohesive” displacements, and, therefore, depend only on l_ω^2 and l_θ^2. Therefore, if kinematic boundary conditions hold (see (12)), then the problem, as a whole, is two-parametric. In the general case of static boundary conditions, only the static factor with $f_a = (4C_2 R_{i,ia} + C_5 \Delta R_a)$ depends on the third parameter C_2. Consequently, the boundary value problem, as a whole, becomes three-parametric only in the case of static nonclassical boundary conditions (12).

Further, if we assume that $\frac{l_\theta^2}{l_\omega^2} = \frac{(2\mu+\lambda)}{\mu}$, $l^2 = l_\omega^2$, then we come to a one-parameter model for which the expansion [14-16, 20,22] takes place: $R_i = U_i - u_i$.

Finally, we note that the fulfillment of the hypothesis of "classicality", in which the static boundary conditions on the tensor of “classical” stresses have the standard classical form (16), generally leads to the possibility of constructing approximate solutions of a wide class of applied problems with the decrease of order of boundary value problems.

Suppose that there are boundary value problems containing the static boundary condition (16) as one of the boundary conditions on the body surface. We will assume at the first step of constructing an approximate solution, that for the tensor of stresses τ_{ij} the defining relation can be approximately written in the form $\tau_{ij} = C_{ijmn}U_{m,n}$. Then the displacement vector U_i can be found from the solution of the first classical boundary-value problem (a problem with static boundary conditions). At the final step, the solution of the boundary value problem for equation (22) is constructed

$$\{(...)\delta_{jk} - l_\omega^2\Delta(...)\delta_{jk} - (l_\theta^2 - l_\omega^2)(...)_{,jk}\}R_k = U_j,$$

with boundary conditions defined by the variational equality

$$\oiint \{[4(C_1+C_2)R_{m,ma}+(4C_2+C_5)\Delta R_a]n_a\delta\theta-2(4C_2R_{i,ia}+C_5\Delta R_a)\delta(\omega_k n_j \text{Э}_{kja})\}dF=0$$

Then the field of "cohesive" displacements from equality (25) can be explicitly determined. After that, we can redefine the stresses in formula (33), assuming that the field of "cohesive" displacements is known, and repeat the procedure for constructing the solution, which reduces to a sequence of solving two boundary value problems of second and not fourth order. It is not difficult to see that the algorithm proposed above corresponds to the procedure for constructing a solution using the asymptotic expansion of the solution for a small parameter $l^2=l_\omega^2$ and resembles the procedure for the method of elastic solutions. In this case, the equilibrium equations (15) and static boundary conditions (16) are satisfied exactly at each step, and the defining relations are considered as approximate, which is completely permissible.

4. Applied "vector" gradient models

For applied problems, the simplest gradient models that contain two or even one additional parameter are of interest, comparing with the classical theory of elasticity. Let's consider some variants of such correct "vector" gradient models.

Suppose that in (10) $C=0$. In the future, we will use the same transformations for model analysis as we used in the section 3. The variational equation of the applied gradient two-parameter model in this case has the form:

$$\begin{aligned}\delta L=&\iiint \{C_{ijmn}R_{m,nj}-C_5\Delta\Delta R_i-4C_2(1+\frac{C_2}{C_5})\Delta R_{j,ji}+P_i^V\}\delta R_i dV+\\
&+\oiint \{P_i^F-[C_{ijmn}R_{m,n}-C_5\Delta R_{i,j}-C_2\Delta R_{j,i}-C_2(1+2\frac{C_2}{C_5})\Delta R_{k,k}\delta_{ij}-\\
&-2C_2(1+\frac{C_2}{C_5})R_{m,mij}]n_j\}\delta R_i dF+\\
&+\oiint \{-C_5(\Delta R_a+2\frac{C_2}{C_5}R_{m,ma})\delta[R_{a,j}n_j+\frac{C_2}{C_5}(n_aR_{j,j}+n_kR_{k,a})]\}dF=0.\end{aligned} \tag{34}$$

For the model (34), the "classical" equilibrium equations and the "classical" static boundary conditions (with variation of displacements δR_i in (34)) have a clearly classical form:

$\tau_{ij,j}+P_i^V=0$, $\oiint (P_i^F-\tau_{ij}n_j)\delta R_i dF=0,$

where τ_{ij} is the tensor of "classical" stresses:

$$\tau_{ij}=C_{ijmn}R_{m,n}-C_5\Delta R_{i,j}-C_2\Delta R_{j,i}-C_2(1+2\frac{C_2}{C_5})\Delta R_{k,k}\delta_{ij}-2C_2(1+\frac{C_2}{C_5})R_{m,mij}. \tag{35}$$

The stresses (35), in contrast to (14), can be made paired, requiring in addition: $C_2=C_5$.

The nonclassical boundary conditions in (34) decompose into three pairs of alternative nonclassical boundary conditions:

$$\oiint \{-C_5(\Delta R_a+2\frac{C_2}{C_5}R_{m,ma})\delta[R_{a,j}n_j+\frac{C_2}{C_5}(n_aR_{j,j}+n_kR_{k,a})]\}dF=0. \tag{36}$$

The "vector" (three-parameter) theory (12), (17) differs from the theory of the "cohesive" field (two-parametric) model (34), (36) in that the boundary conditions contain all three nonclassical parameters.

For the model under consideration, the operator of the equilibrium equation is represented as the product of a classical equilibrium operator and an additional, nonclassical Helmholtz operator if the scale parameters l_θ^2, l_ω^2 in (24) are related to nonclassical modules by the following relations:

$$\begin{cases} C_5 = \mu l_\omega^2 \\ 2C_2 = \sqrt{\mu(2\mu+\lambda)} l_\omega l_\theta - \mu l_\omega^2 \end{cases}. \qquad (38)$$

The "classical" displacement field, the field of "cohesive" displacements in this model, is also determined by equations (22) and (25), and the general solution is represented by the relation (32).

Let us give one more particular "vector" gradient model, which is a further simplification of the general vector model and is already a one-parameter gradient model. We assume in (11), (14) that $C = 0$, $C_5 = \mu l^2$, $2C_2 = (\mu+\lambda) l^2$. Then the density of the gradient part of the potential energy can be represented in a simpler and more compact form:

$$C_{ijkmnl} R_{i,jk} R_{m,nl} = \mu l^2 [(\Delta R_a - R_{i,ia}) + \frac{(2\mu+\lambda)}{\mu} R_{i,ia}][(\Delta R_a - R_{i,ia}) + \frac{(2\mu+\lambda)}{\mu} R_{j,ja}]. \qquad (37)$$

Here it is taken into account that

$$C_{ijkmnl} = \mu l^2 [(\delta_{jk}\delta_{ia}) + \frac{(\mu+\lambda)}{2\mu}(\delta_{ij}\delta_{ka} + \delta_{ik}\delta_{ja})][(\delta_{nl}\delta_{ma}) + \frac{(\mu+\lambda)}{2\mu}(\delta_{mn}\delta_{la} + \delta_{ml}\delta_{na})].$$

For this particular model (37), the variational equation defining the mathematical model (solving the equation and the boundary conditions) has the form:

$$\begin{aligned} \delta L = \\ = \iiint \{\tau_{ij,j} + P_i^V\}\delta R_i dV + \oiint \{P_i^F - (P_i^F - \tau_{ij} n_j)\}\delta R_i dF + \\ - \oiint l^2 [\mu \Delta R_a + (\mu+\lambda) R_{m,ma}] \delta [R_{a,k} + \frac{(\mu+\lambda)}{2\mu}(R_{j,j}\delta_{ak} + R_{k,a})] n_k dF = 0, \end{aligned} \qquad (38)$$

where τ_{ij} are the "classical" stresses:

$$\begin{aligned} \tau_{ij} = C_{ijmn} R_{m,n} - l^2 [\mu \Delta R_{i,j} + \frac{(\mu+\lambda)}{2} \Delta R_{j,i} + \\ + (\mu+\lambda) \frac{(3\mu+\lambda)}{2\mu} R_{m,mij} + \frac{(\mu+\lambda)}{2\mu}(2\mu+\lambda)\Delta R_{k,k}\delta_{ij}]. \end{aligned}$$

If we assume $l_\theta^2 = (2\mu+\lambda) l^2 / \mu$, $l^2 = l_\omega^2$ then for the one-parameter model (38) under consideration, the operator of the equilibrium equation is represented as the product of the Lame operator and the generalized Helmholtz operator constructed on the base of the Lame operator (see also [20]):

$$[L_{ij}(...)]\{(...)\delta_{jk} - (l^2/\mu) L_{ij}(...)_{,jk}\} R_k + P_i^V = 0, \qquad (39)$$

where $L_{ij}(...)$ is the Lame operator, $L_{ij}(...) = [\Delta(...)\delta_{ij} + (\mu+\lambda)(...)_{,ij}]$.

The "classical" displacement field and the field of "cohesive" displacements are determined, respectively, by the equalities:

$$U_j = R_j - [\mu \Delta R_j + (\mu+\lambda) R_{k,kj}] l^2/\mu, \quad u_j = -(l^2/\mu) L_{jk} R_k \qquad (40)$$

and are the solutions of equations:

$$\begin{aligned} [\mu\Delta(...)\delta_{ij} + (\mu+\lambda)(...)_{,ij}] U_j + P_i^V = 0 \\ L_{ij} u_j - (\mu/l^2) u_i + P_i^V = 0 \end{aligned} \qquad (41)$$

The general solution is represented as a decomposition: $R_i = U_i - u_i$.

Note that the one-parameter gradient model for which the equalities (39) - (41) are satisfied was widely used in [19, 20, 22] to solve applied problems in the mechanics of composites with micro/nano-dimension inclusions and was called the applied model of the interphase layer.

5. On one generalization of the Aifantis's GradEla model

Finally, we consider an even more particular gradient model, which belongs to the class of vector models.

We suppose that $C_1 = 0, C_2 = 0, C_5 = \mu l^2$. Then the relation (11), (14) gives the following representation for the tensor of gradient modules $C_{ijkmnl} = \mu l^2 (\delta_{jk}\delta_{ia})(\delta_{nl}\delta_{ma})$, and the gradient part of the potential energy has the form:

$$C_{ijkmnl} R_{i,jk} R_{m,nl} = \mu l^2 \Delta R_a \Delta R_a \,. \tag{42}$$

The variational equation of the vector gradient model under consideration looks like:

$$\begin{gathered}\delta L = \delta A - \iiint [C_{ijmn} R_{m,n} \delta R_{i,j} + C_{ijkmnl} R_{m,nl} \delta R_{i,jk}] dV = \\ = \oiint \{[P_i^F - (C_{ijmn} R_{m,n} - \mu l^2 \Delta R_{i,j}) n_j] \delta R_i - \mu l^2 \Delta R_a \delta(R_{a,k} n_k)\} dF = 0\end{gathered} \tag{43}$$

It follows from the variational equality (43) that in the boundary-value problem the “classical” static condition for the “classical” stress τ_{ij} is precisely distinguished, and three pairs of alternative nonclassical boundary conditions are given by the variational equality:

$$\oiint \mu l^2 \Delta R_a \delta(R_{a,k} n_k) dF = 0.$$

In this case, the “classical” stress has the form

$$\tau_{ij} = C_{ijmn} R_{m,n} - \mu l^2 \Delta R_{i,j} \,, \tag{44}$$

and, in its structure, almost exactly coincides with the expression for the total stresses of the GradEla model of Aifrantis.

It is easy to verify that the equilibrium equation for a given vector model exactly coincides with the equilibrium equation of the GradEla model, and the operator of the equilibrium equation is represented as the product of the Lame operator and the Helmholtz operator

$$\{(...) - l^2 \Delta(...)\}[\mu \Delta(...) \delta_{ik} + (\mu + \lambda)(...)_{,ik}] R_k + P_i^V = 0. \tag{45}$$

The “classical” displacement field U_j and the "cohesive" displacement field u_j are determined by the equations:

$$U_j = R_j - l^2 \Delta R_j \,, \quad [\mu \Delta(...) \delta_{ij} + (\mu + \lambda)(...)_{,ij}] U_j + P_i^V = 0 \tag{46}$$

$$u_i = -[\Delta(...) \delta_{ik} + (...)_{,ik} (\mu + \lambda) / \mu] R_k \,, \ \mu l^2 \Delta u_i - \mu u_i + P_i^V = 0 \,, \tag{47}$$

which also coincide exactly with the corresponding equations of the Aifantis GradEla model [36].

Note that although the gradient model determined by the relations (42) - (47) resembles the gradient model of Aifantis (GRADELA) in many ways, does not coincide with it. The model presented above is non-symmetric – the “classical” stresses are non-symmetric. In the Aifantis model, the gradient part of the potential energy is written in the form $\mu l^2 \varepsilon_{ij,k} \varepsilon_{ij,k}$ and differs from the expression (42), the gradient component of the defining relation for symmetric total stresses is written through Laplacian of the deformation $\Delta \varepsilon_{i,j}$, in contrast to expression (44).

The model considered in the article belongs to the class of vector gradient correct models. For it, the static boundary condition, written only for “classical” stresses, is precisely distinguished. In general, this leads to simplifying the construction of solutions of applied problems. The GradEla model of Aifantis does not possess this quality. It does not belong to the class of vector gradient correct models.

The model defined by (42) - (47) we will call the generalized Aifantis model. This generalization allows us to transfer the Aifantis model to a class of correct vector models for which the classical boundary.

6. On decomposition of boundary value problems

Let us return to the vector gradient models and briefly examine the possibility of substantially simplifying the solutions of boundary value problems for them using **decomposition** of the general boundary value problem of the fourth order into a sequence of independently solvable boundary value problems of the second order. We assume that the conditions that lead to static boundary conditions of the classical form are satisfied:

$$C_{ijkmnl} = \mu l_{akij} l_{almn}. \tag{48}$$

Then the following statement holds: The gradient part of the potential energy density for the model in which the gradient modules obey conditions (48) is representable as the potential energy density of vector field. Really, taking into account (48) we obtain:

$$C_{ijkmnl} R_{i,jk} R_{m,nl} = \mu l_{akij} l_{almn} R_{i,jk} R_{m,nl} = \mu (l_{akij} R_{i,jk})(l_{almn} R_{m,nl}). \tag{49}$$

The expression (49) is determined by the convolution of the following vectors $\varepsilon_i = l_{akij} R_{i,jk}$. Consequently, for the gradient models under consideration, the variational equation, taking into account (48), (49) takes the form:

$$\begin{aligned} \delta L = \iiint [(C_{ijmn} R_{m,n} + \mu \varepsilon_{a,k} l_{akij})_{,j} + P_i^V] \delta R_i dV + \\ + \oiint \{[P_i^F - (C_{ijmn} R_{m,n} + \mu \varepsilon_{a,k} l_{akij}) n_j] \delta R_i - \mu \varepsilon_a \delta (l_{akij} n_k R_{i,j})\} dF = 0. \end{aligned} \tag{50}$$

We can define the second rank tensor in (50) as the tensor of conditional “classical” stresses:

$$\sigma_{ij} = (C_{ijmn} R_{m,n} + \mu \varepsilon_{a,k} l_{akij}). \tag{51}$$

It is easy to see that the stresses σ_{ij} (51) satisfy both the equilibrium equations and the classical static conditions:

$$\begin{aligned} \delta L = \iiint (\sigma_{ij,j} + P_i^V) \delta R_i dV + \\ + \oiint [(P_i^F - \sigma_{ij} n_j) \delta R_i - \mu \varepsilon_a \delta (l_{akij} n_k R_{i,j})] dF = 0. \end{aligned} \tag{52}$$

The variational equation (52) indicates that nonclassical conditions are determined by three pairs of alternative boundary conditions which do not change the classical boundary conditions $P_i^F - \sigma_{ij} n_j = 0$.

Using the classical modulus of elasticity $C_{ijmn} = \lambda \delta_{ij} \delta_{mn} + \mu(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm})$, let's postulate the following relations:

$$l_{ijmn} = \frac{l}{\mu} C_{ijmn} = l(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm} + \frac{\lambda}{\mu}\delta_{ij}\delta_{mn}), \quad l_{ijmn} = l_{mnij} \tag{53}$$

Therefore, taking into account (49) we find that the following equality must hold:

$$C_{ijkmnl} = \frac{l^2}{\mu} C_{akij} C_{almn}, \qquad C_{almn} = \lambda \delta_{al} \delta_{mn} + \mu(\delta_{am}\delta_{ln} + \delta_{an}\delta_{lm}).$$

In this case, the relation (51) takes the form:

$$\sigma_{ij} = (C_{ijmn} R_{m,n} + \mu \varepsilon_{a,k} l_{akij}) = C_{ijmn} (R_m + l \varepsilon_m)_{,n}. \tag{54}$$

We note, however, that the introduction of hypothesis (53) leads to a loss of symmetry for the “classical” stresses (54).

At last, “classical” displacements U_i can be found (see eq. (53)):

$$U_i = R_i + l\varepsilon_i = R_i + l \cdot l_{ilmn} R_{m,nl} = R_i + l^2 (\delta_{im}\delta_{ln} + \delta_{in}\delta_{lm} + \frac{\lambda}{\mu}\delta_{il}\delta_{mn}) R_{m,nl} =$$
$$= R_i + [\mu \Delta R_i + (\mu+\lambda) R_{k,ki}] l^2 / \mu. \tag{55}$$

"Cohesive" displacements u_i are defined through the difference between "classical" and total displacements [20, 22]:

$$u_i = \frac{l^2}{\mu}[\mu\Delta R_i + (\mu+\lambda)R_{k,ki}]. \tag{56}$$

Then general solution for the considered variant of the "vector" gradient model has the form:

$$R_i = U_i - u_i \tag{57}$$

Note, that the gradient model defined by equations (50)-(57) is unique one parametrical model which allow to simplify set of boundary volume problems using the decompositions of the initial problems of fourth order to the sequence of two problems of second order.

As a result, for the "vector" gradient model, the first fundamental problem splits into two, the classical boundary value problem:

$$\begin{cases} C_{ijmn} U_{m,nj} + P_i^V = 0 \\ \oiint (P_i^V - C_{ijmn} n_j U_{m,n}) \delta(U_i - u_i) dF = 0 \end{cases}, \tag{58}$$

and the auxiliary boundary value problem:

$$\begin{cases} (l^2/\mu) C_{ijmn} R_{m,nj} - R_i = -U_i \\ \oiint (R_i - U_i) \delta(C_{ijmn} n_j R_{m,n}) dF = 0 \end{cases}.$$

The decomposition of the general solution into a superposition of "classical" one and "cohesive" one leads to the fact that the boundary value problems of gradient theories, in some cases, can be represented as a sequence of solutions of two boundary value problems: classical, with respect to the vector of "classical" displacements U_i and the boundary value problem with respect to the vector of complete displacements R_i. The non-classical auxiliary to (58) the boundary value problem can be reformulated, in accordance with (57) with respect to "cohesive" displacements u_i:

$$\begin{cases} C_{ijmn} u_{m,nj} - (\mu/l^2) u_i + P_i^V = 0 \\ \oiint u_i \delta C_{ijmn} n_j (U_{m,n} - u_{m,n}) dF = 0. \end{cases} \tag{59}$$

Consequently, for the first fundamental problem, the boundary value problems always disintegrate into "classical" and "cohesive" displacements for the "vector" gradient model under consideration.

7. Analysis and decompositions of the boundary value problems

Formally, the boundary value problems of the "vector" gradient model, in the general case, are coupled problems (58), (59):

$$\begin{cases} C_{ijmn} U_{m,nj} + P_i^V = 0 \\ \oiint (P_i^V - C_{ijmn} n_j U_{m,n}) \delta(U_i - u_i) dF = 0 \end{cases} \quad \begin{cases} C_{ijmn} u_{m,nj} - (\mu/l^2) u_i + P_i^V = 0 \\ \oiint u_i \delta(C_{ijmn} n_j (U_{m,n} - u_{m,n})) dF = 0 \end{cases}. \tag{60}$$

For definiteness, we will assume that in the surface integral the multiplier associated with variation determines as the "static factors" in the boundary conditions, and the expression under the variation determines as "kinematic factors". Let us consider four basic formulations of boundary value problems for statements (60).

1. For “classical” and “cohesive” displacements, it is required to perform static boundary conditions:

$$(P_i^V - C_{ijmn} n_j U_{m,n}) = 0, \qquad u_i = 0. \tag{61}$$

In this case, we can see that the boundary value problems (60), (61) with respect to vectors of “classical” U_i and “cohesive” u_i displacements are separated by their construction.

2. For “classical” displacements, the static boundary conditions are satisfied, and for “cohesive” displacements, are performed the kinematic boundary conditions:

$$(P_i^V - C_{ijmn} n_j U_{m,n}) = 0, \qquad \delta(C_{ijmn} n_j U_{m,n} - C_{ijmn} n_j u_{m,n}) = 0. \tag{62}$$

Varying $(P_i^V - C_{ijmn} n_j U_{m,n}) = 0$ and adding up with $\delta(C_{ijmn} n_j U_{m,n} - C_{ijmn} n_j u_{m,n}) = 0$, we obtain using (62):

$$(P_i^V - C_{ijmn} n_j U_{m,n}) = 0, \qquad \delta(P_i^V - C_{ijmn} n_j u_{m,n}) = 0. \tag{63}$$

As a result we again receive the full decomposition of the boundary value problems (60),(63) for the vectors of “classical” U_i and “cohesive” u_i displacements.

3. For “classical” displacements, kinematic boundary conditions are performed, and for “cohesive” displacements, the “static” boundary conditions are satisfied:

$$\delta(U_i - u_i) = 0, \qquad u_i = 0. \tag{64}$$

Varying $u_i = 0$ and adding up with $\delta(U_i - u_i) = 0$, we obtain from (64) the following boundary conditions:

$$\delta U_i = 0, \qquad u_i = 0. \tag{65}$$

Conditions (65) lead to fully decomposition of the boundary value problems for the vectors of “classical” U_i and “cohesive” u_i displacements.

4. The kinematic boundary conditions are satisfied for the “classical” displacements and for “cohesive” displacements:

$$\delta(U_i - u_i) = 0, \qquad \delta(C_{ijmn} n_j (U_{m,n} - u_{m,n})) = 0. \tag{66}$$

The boundary conditions (66) do not allow to divide boundary volume problems respect to vectors of “classical” U_i and “cohesive” u_i displacements. Indeed, since $C_{ijmn} = C_{ijmp}\delta^*_{pn} + (C_{ijmp} n_p) n_n$ and $C_{ijmp}\delta^*_{pn}\delta(U_i - u_i)_{,n} = 0$, from the second condition (66) follows that $\delta(U_{i,j} n_j - u_{i,j} n_j) = 0$. Thus, the boundary-value problem for the vectors of “classical” U_i and “cohesive” u_i displacements takes the form:

$$\delta(U_i - u_i) = 0, \qquad \delta(U_{i,j} n_j - u_{i,j} n_j) = 0. \tag{67}$$

Conditions (66) (and (67)) define the coupled boundary value problem respect to vectors of “classical” U_i and “cohesive” u_i displacements.

8. Conclusions

It is shown that the traditional formulation of gradient theories of elasticity leads to the fact that classical static conditions on the surface of the body are not satisfied locally. A "vector" theory is formulated, it is correct and provides a classical view of static boundary conditions.

Particular cases of vector gradient models are considered and it is shown that there exists a particular vector gradient model whose equilibrium equations coincide with the equations of the well-known applied GradEla model of Aifantis. Such a vector gradient model can be considered as a generalization of the Aifantis model. For it there is an exact decomposition of static boundary conditions to "classical" stresses (full stresses if we use Aifantis definition). Finally, it is shown that if we neglect the symmetry requirement for the gradient-module tensor with respect to the last indices in triples, then it is possible to indicate a unique gradient theory that admits the decomposition of boundary value problems of the fourth order into a sequence of two second-order boundary value problems when we solve a number of boundary value problems.

Acknowledgements. *This work was carried out with support from MegaGrant Project No. 14.Z50.31.0039 in Togliatti State University*

References

[1] R. D. Mindlin // *Archive for Rational Mechanics and Analysis* **16** (1964) 51.
[2] R. D. Mindlin, N. N. Eshel // *International Journal of Solids and Structures* **4** (1968) 109.
[3] E.C. Aifantis // Int. J. Engng. Sci. **30** (1992) 1279.
[4] B. Altan, E. C. Aifantis // Scr.Metall.Mater. **26** (1992) 319.
[5] C. Ru, E. Aifantis // *Acta Mechanica*. **101(1)** (1993) 59.
[6] B. S. Altan, E. C. Aifantis // *Journal of the Mechanical Behavior of Materials **8*** (1997) 231.
[7] S. Lurie, P. Belov // Engineering Fracture Mechanics **130** (2014) 3.
[8] S.A. Lurie, P.A. Belov // *Int. J. Fract*. **50** (2008) 181.
[9] M. Y. Gutkin, E. C. Aifantis // *Scripta Materialia* **35**(1996) 1353.
[10] M. Y. Gutkin, E. C. Aifantis //*Scripta Materialia* **36** (1997) 129.
[11] M. Lazar, D. Polyzos // *International Journal of Solids and Structures* **62**(2015) 1.
[12] H. Askes, A. V. Metrikine // *European Journal of Mechanics-A/Solids* **21**(2002) 573.
[13] H. Askes, E. C. Aifantis // *Physical Review B* **80** (2009) 195412.
[14] S.A. Lurie, P.A. Belov, N. P. Tuchkova // *Int. J. Comp. Mater. Sci*. **36** (2005) 145.
[15] S.A. Lurie, P.A. Belov, D.B. Volkov-Bogorodsky, N.P. Tuchkova // *J. Mat. Sci*. **41** (2006) 6693.
[16] S.A. Lurie, D.B. Volkov-Bogorodsky, V.I. Zubov, N.P. Tuchkova // *Int. J. Comp. Mater. Sci*. **45** (2009) 709.
[17] G.A. Maugin, A.V. Metrikine. In: *One hundred years after the Cosserats*, 1st ed. (Glasgow, Scotland, UK, 2010) Vol. 21, p. 223.
[18] H. M. Ma, X.-L. Gao // *Acta Mechanica* **225** (2014) 1075.
[19] D. B. Volkov-Bogorodskii, S. A. Lurie //*Mechanics of Solids* **51** (2016) 161
[20] S. Lurie, D. B. Volkov-Bogorodsky, A. Leontiev, E. Aifantis // *International Journal of Engineering Science* **49** (2011) 1517.
[21] A. A. Gusev, S. A. Lurie // *Advanced Engineering Materials* **12** (2010) 529.
[22] S. Lurie, D. Volkov-Bogorodskii, N. Tuchkova // *Acta Mechanica* **227** (2016) 127.
[23] D. C. C. Lam, F. Yang, A. C. M. Chong, J. Wang, P. Tong // *Journal of the Mechanics and Physics of Solids* **51** (2003) 1477.
[24] C. Liebold, W. H. Müller // *Computational Materials Science* **116** (2016) 52.
[25] R. Maranganti, P. Sharma // *Journal of the Mechanics and Physics of Solids **55*** (2007) 1823.

[26] R. Barretta, M. Brčić, M. Čanađija, R. Luciano, F. Marotti de Sciarra // *European Journal of Mechanics, A/Solids* **65** (2017) 1.
[27] H. M. Shodja, A. Zaheri, A. Tehranchi // *Mechanics of Materials* **61** (2013) 73.
[28] J.W. Hutchinson, T.Y. Wu, In: *Advances in Applied Mechanics* (Academic Press, New York, 1997), Vol.33, pp.295.
[29] A. A. Gusev, S. A. Lurie // *Mathematics and Mechanics of Solids* **22** (2015) 1.
[30] H. Askes, E. C. Aifantis // *International Journal of Solids and Structures* **48** (2011) 1990.
[31] C. A Polizzotto // *European Journal of Mechanics / A Solids* **7** (2016) 261.
[32] P. A.Belov, S.A.Lurie. *Mathematical theory of defectness media. Gradient theories, Formulations, hierarchy. Analysis. Applications* (Palmarium Academic Publishing, 2014), pp. 337.
[33] A. Jafari, S. Shirvani Shah-Enayati, A. A. Atai // *European Journal of Mechanics, A/Solids* **59** (2016) 124.
[34] S. A. Lurie, E. L. Kuznetsova, L. N. Rabinskii, E. I. Popova // *Mechanics of Solids* **50** (2015) 135.
[35] S. A. Lurie, D. B. Volkov-Bogorodsky, P. A. Belov, E. D. Lykosova// *Nanomechanics Science and Technology: An International Journal* **7** (2016) 261.
[36] X. L. Gao, S. K. Park // *International Journal of Solids and Structures* **44** (2007) 7486
[37] M. Lazar, D. Polyzos //*International Journal of Solids and Structures* **62**(2015) 1.
[38] Elias C. Aifantis // *Theoretical and Applied Mechanics Letters* **4** (2014) 1.