Materials Physics and Mechanics 32 (2017) 370-382 Received: November 18, 2017

COMPARISON OF ADAPTIVE ALGORITHMS FOR SOLVING PLANE PROBLEMS OF CLASSICAL AND COSSERAT ELASTICITY

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Abstract. Paper continues developments and numerical testing of functional approach [1-3] to a posteriori error control for 2D problems of classical [2] and Cosserat elasticity [4,5]. The approach yields reliable error bounds (majorants) that are valid for all conforming solutions of problems regardless of methods used for a numerical implementation of a solution process. Efficiency of the above technique is shown on a set of numerical examples including consequent mesh adaptations with MATLAB tools as it was done [6].
Keywords: computational mechanics; a posteriori error estimates; finite element method.

1. Introduction

Various boundary-value problems of classical elasticity theory have been intensively used for developments, numerical testing and comparison of different approaches to a posteriori error control. Such methods are aimed to explicitly compute some quantitative measure of errors, which appear during numerical simulations, and indicate subdomains with large errors for further refinements. All general frameworks for error estimation and adaptive mesh refinement have been applied to linear elasticity. The first theoretical result appeared in [7] (much earlier than others, like [8-10]). W. Prager and J.L. Synge considered a “geometrical” method of error estimation based on originally intuitive constructions. But this idea gave a rise to another approach of P. Ladevèze and colleagues (see [11-14] for reviews). It is based on the concept of *errors in constitutive relations* or CRE. However, [13] shows that computational efforts to get sharp error estimates with this method can be significant.

Another approach that is widely used nowadays is the so-called *gradient averaging*. It is based on pioneering works of O.C. Zienkiewicz and J.Z. Zhu [9,15]. The last paper includes a comprehensive study of various computational aspects with different examples of implementation of averaging procedures to problems of solid mechanics with different types of finite elements. The main advantage of this method is simplicity, but it isn’t able to provide reliable error control and often underestimates true errors. Another series of famous publications of O.C. Zienkiewicz and J.Z. Zhu appeared in 1992 [16-18] with a new approach called *superconvergent patch recovery* or SPR, which is quite popular nowadays – see [19-23].

Group of *residual-based* methods for linear elasticity started to develop from paper by C. Johnson and P. Hansbo [10], which also includes numerical results for plane strain statement. For further research on explicit and implicit residual methods, we refer to [24-33]. Recent results on residual-type indicators and other methods in application to plane problems of linear elasticity theory one can find, for example, in [27,34,35].

Paper [22] contains a comprehensive study and comparison of various modifications of the SPR-method with the same conclusion about possible underestimation of the true error that yields unreliability of the method. For extended review of the literature, we mention [19] and [36]. In [37] one can find comparison of 6 indicators of different types. Authors of [38] compared CRE and SPR methods (see also [39]). In 1994 paper of I. Babuška and colleagues [40] provided a special methodology for comparison of indicators of different types and presented a review of early results on error estimation theory. Investigation has been continued in [41-43], and in [44] – with adaptations.

It is necessary to note that collection [45] edited by P. Ladevèze and J.T. Oden, and the review by R. Verfürth [46] are also very useful for analysis of various groups of classical methods of a posteriori error control for problems of solid mechanics. Nowadays, the theory of a posteriori error control forms one of the important directions of modern computational mathematics. The amount of the corresponding literature is increasing continuously from the end of 1970-s (see, for instance, [3,47,48] for a review). However, summarizing these results, one can conclude that computationally inexpensive approaches are unreliable, especially in error control of solutions of black-box software for Computer-Aided Engineering (CAE). Some modifications, which increase reliability, may lead to extra computational efforts and rather technical implementations. All standard approaches are based on the fact that controlled numerical solution is an exact solution of a discrete problem generated by Finite Element Methods (FEM). Often, this is not the case for commercial software.

Theoretical background of the functional approach to a posteriori error control, including estimates for various problems of continuum mechanics, has been developed starting from pioneering work of S. Repin and L.S. Xanthis [49]. The early results were mostly theoretical – some references can be found in [2,3,50]. For the last decade, investigations of the functional approach by S. Repin and his colleagues become more practice-oriented. Functional-type a posteriori error majorants for classical linear elastic problems have been obtained in [51] and [2] using two different methodologies.

Cosserat continuum [52] is one of interesting and sufficiently straightforward generalizations of the classical theory (see, for example, [53] and [54] for mathematical statements). Numerical methods for solving problems related to Cosserat continuum began to develop more intensively from the XXI century (see, for example, [55-59]). Nevertheless, first results concerning functional-type error estimates have appeared during the last few years. Totally, there are only few papers addressed to a posteriori error control for computed approximations – [60,61,4,5], and this work requires further developments in construction and comparison of adaptive algorithms.

2. Statement

Majorants for both mathematical models under consideration have some important features in common. Estimates for classical and Cosserat elasticity have the form

$$|||e||| \leq M := D(\tilde{u}, s^*) + R(s^*) + \textit{penalty terms}, \qquad e := u - \tilde{u}, \tag{1}$$

where := means "equality by definition", u contains all components of the exact solution, which is generally unknown, $\tilde{u}$ represents approximations of these components, which are explicitly provided from computations, e is the corresponding error vector formed by components of deviations from exact values, s^* is a set of auxiliary variables, and |||...||| denotes the global (energy) norm of the error. All components of functional-type error majorants have clear physical meaning and interpretation. Term D represents errors in constitutive relations. Term R is a residual term with mesh-independent constants (some proper balance of equilibrium equations). The estimate (1) may contain optional penalty terms that violate the symmetry condition for auxiliary tensors in a weak form. Therefore, the right-hand side of (1), denoted as M, depends only on known data – approximate solutions, constants, positive parameters, additional variables, and it can be calculated explicitly. This estimate is exact in the sense that the equality can be achieved with a proper setting of parameters and variables. For instance, estimates for plane problems of the considered types have the form (1) – see [2,4,61] for details. All auxiliary fields can be constructed on a common basis of finite elements suitable for space H(div) – the Hilbert space of square summable vector-functions with square summable divergence.

A reasonable choice of approximations for free variables in functional-type error estimates allows obtaining accurate guaranteed upper bounds of errors. The functional approach does not impose significant additional restrictions (for example, the assumption about exact satisfaction of equilibrium equations) on free variables. A functional-type error estimate is applicable to any arbitrary approximate solution from the corresponding energy space. It remains valid regardless of the approach used for calculating this solution, thus it allows taking into account various error sources, what is extremely important for additional verification of commercial software for CAE. Additionally to the global error estimation procedure, the functional M^2 can be split and used as an indicator of the local error distribution, considering the contributions to the global error on each finite element. Therefore, it can provide a basis for construction of adaptive algorithms.

Adaptive algorithms for FEM generally consist of four main steps: solve, estimate, mark and refine (see, for example, [62,63]). Concerning the estimate (1) the procedure admits the following interpretation:

1. *step(solve)*: compute $\tilde{u}$ on some (initial or consequent) finite element mesh;
2. *step(estimate):* compute the functional M from all individual contributions to it on every element;
3. *step(mark):* mark mesh elements with comparatively large local errors by some marking strategy (using some error threshold or percent of the total amount of elements);
4. *step(refine):* divide marked elements and do local mesh refinements.

Besides of local refinements, for more sophisticated and efficient algorithms one can consider some procedures for local mesh coarsening.

3. Numerical results

Adaptive algorithms for plane elasticity problems, mentioned in this paper, are implemented in MATLAB. In the continuation of previous research the mixed-FEM approximations [64,65] are used for computation of upper error bounds and indicators. Extending results of [66], below we consider two examples as an illustration.

For both examples, all material properties are taken from [56].

Example 1 (square domain with a hole). We consider the square domain with side 16.2 mm, which contains a circular hole with radius 0.216 mm in the center. The left edge is fully clamped and the tensile loading of 1 MPa is applied to the right edge. Initial mesh is shown in Fig. 1 (a).

Two types of problems are solved – with classical and Cosserat elasticity models. The resulting adaptive meshes are compared. Results for classical elasticity are collected in Table 1, and for Cosserat model – in Table 3. The lowest-order Raviart-Thomas approximation [64] is used for the implementation of the majorant M from (1).

For this example results were partially presented in [66] with minor modifications of computational algorithms. For instance, final mesh for the majorant for the classical elasticity now consists of 2960 nodes instead of 2955 in [66].

The first block of results in each table corresponds to the uniform mesh refinement with no adaptation. The initial mesh (first column) is provided by a standard MATLAB tool and remains the same for all refinement algorithms. In any uniform refinement step, each element from previous mesh is divided into four new elements. The nodes, elements and relative errors are collected in corresponding table rows. Relative errors are computed with the so-called *reference solution* – an approximate solution obtained on a fine mesh. It is very timeconsuming to calculate the reference solution; therefore, it is provided only for numerical experiments on validation and comparison of different approaches. For engineering practice, it is never used. But the following results show that functional type error majorants can be considered as a reasonable alternative choice.

Table 1. Example 1. Classical elasticity: results for uniform and adaptive mesh refinements.

Uniform refinement					
MESH	1	2	3	4	5
NODES	295	1147	4522	17956	71560
ELEMENTS	557	2228	8912	35648	142592
RELATIVE ERROR, %	10.1	6.6	4.2	2.6	1.6
Reference indicator					
NODES	295	353	423	765	2050
ELEMENTS	557	664	793	1428	3906
RELATIVE ERROR, %	10.1	6.9	4.9	2.6	1.6
Majorant-based indicator					
NODES	295	323	536	876	2960
ELEMENTS	557	606	1002	1648	5701
RELATIVE ERROR, %	10.1	7.1	3.7	2.7	1.4
$I_{eff} = M/\|\|\|e\|\|\|$	1.2	1.2	1.3	1.3	1.2

Table 2. Example 1. Classical elasticity: results for another uniform refinement.

Uniform refinement (another initial mesh)					
MESH	1	2	3	4	5
NODES	305	1183	4658	18484	73640
ELEMENTS	573	2292	9168	36672	146688
RELATIVE ERROR, %	7.5	4.6	2.8	1.7	1.0

In addition, the uniform refinement procedure is repeated from another slightly different initial mesh (Fig. 1 (b)). Results are collected in Table 2. If the desired relative error level is less or equal to 2%, then for the first uniform sequence the resulting mesh contains 71560 nodes, and for the second one a solution process yields the mesh with 18484 nodes only. Thus, choice of the initial mesh may dramatically affect the uniform refinement results and may increase computational costs caused by necessity to provide accurate results.

Table 3. Example 1. Cosserat elasticity: results for uniform and adaptive mesh refinements.

Uniform refinement					
MESH	1	2	3	4	5
NODES	295	1147	4522	17956	71560
ELEMENTS	557	2228	8912	35648	142592
RELATIVE ERROR, %	12.0	9.2	6.6	4.4	2.7
Reference indicator					
NODES	295	348	469	899	2582
ELEMENTS	557	652	870	1668	4899
RELATIVE ERROR, %	12.0	9.8	6.8	4.5	2.8
Majorant-based indicator					
NODES	295	317	527	1039	4111
ELEMENTS	557	592	956	1894	7674
RELATIVE ERROR, %	12.0	9.6	7.0	4.7	2.6

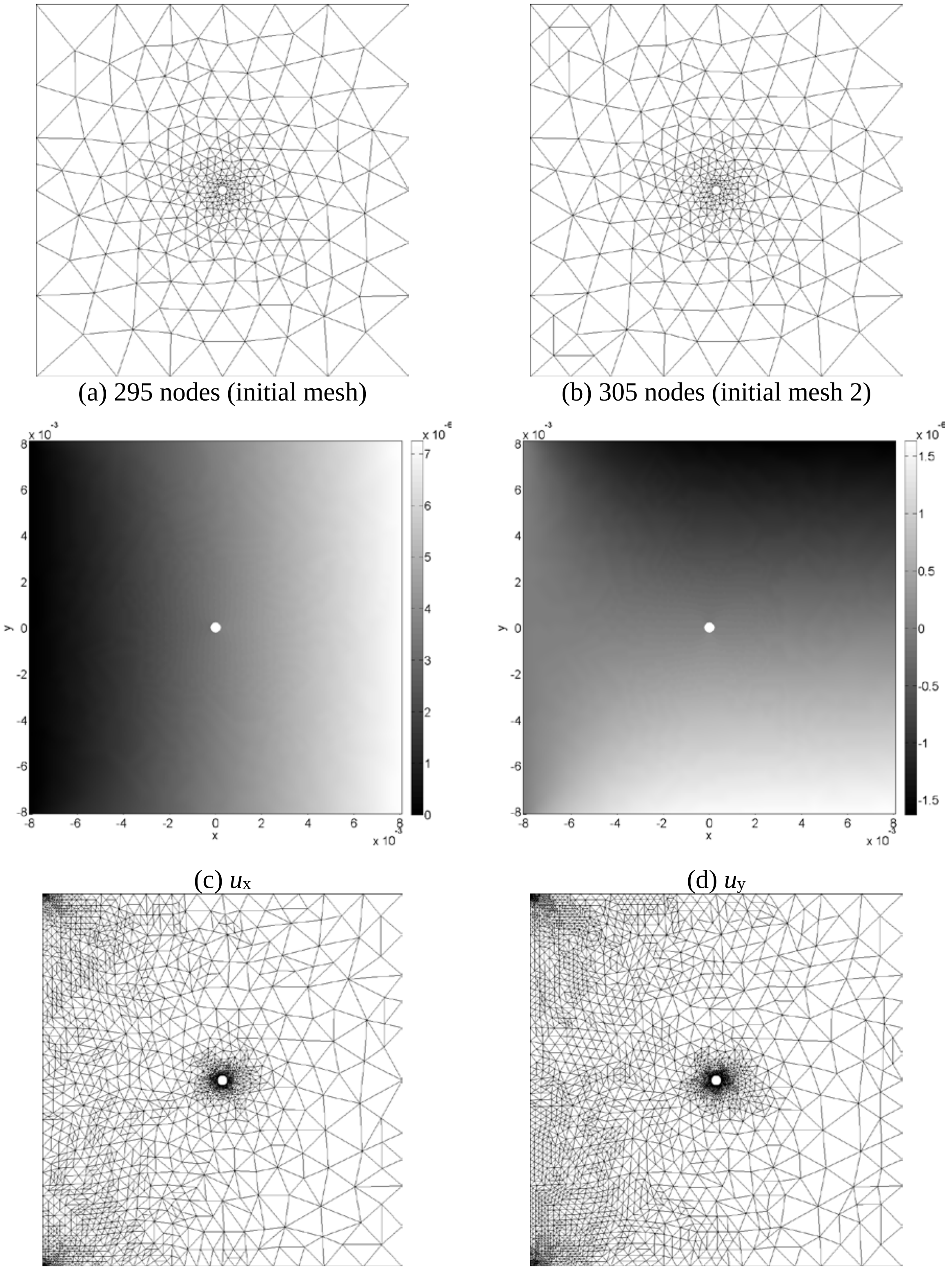
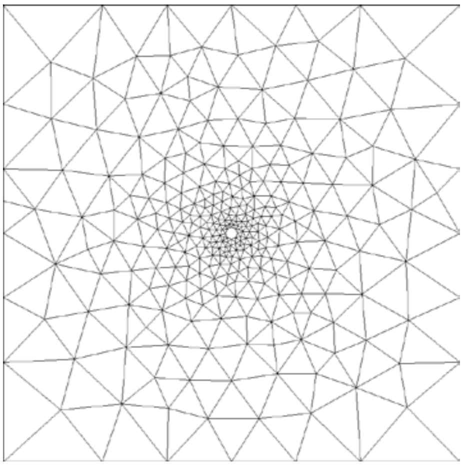
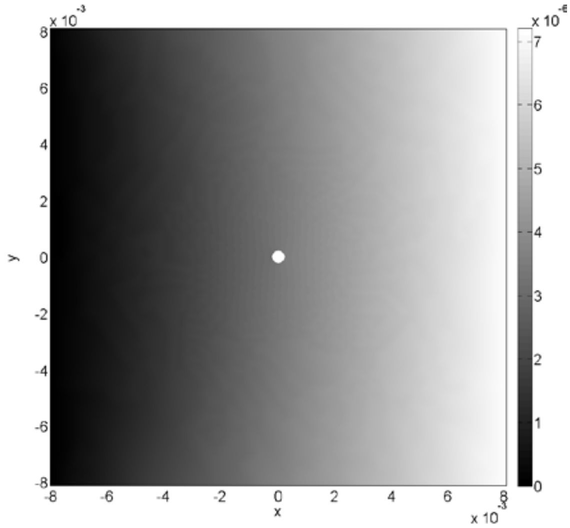


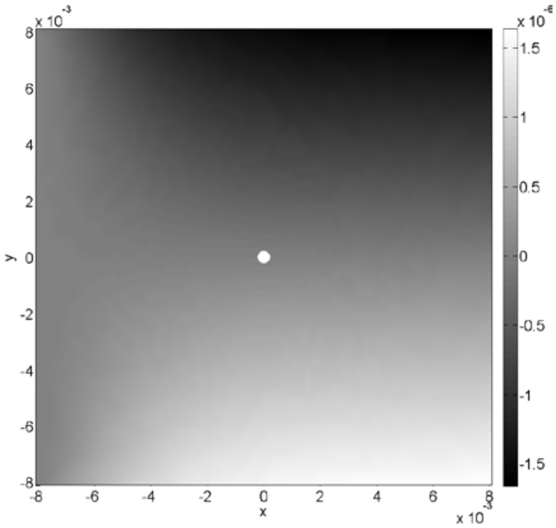
Fig. 1. Example 1. Classical elasticity: initial meshes (a,b), components of the solution u (c,d) (displacements), the result of adaptation by the reference indicator (e), the result of adaptation by majorant-based indicator (f).



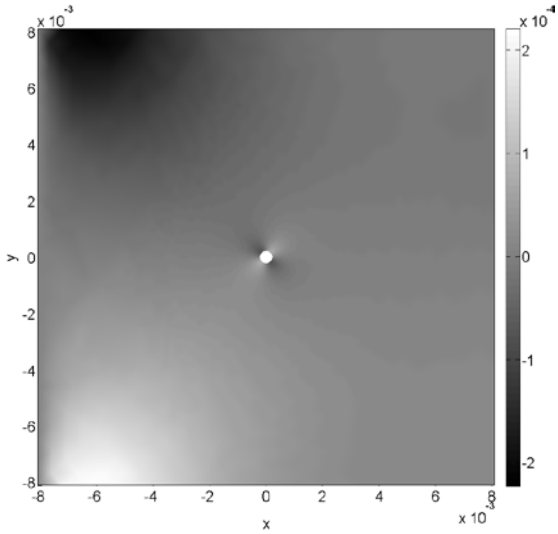
(a) 295 nodes (initial mesh)



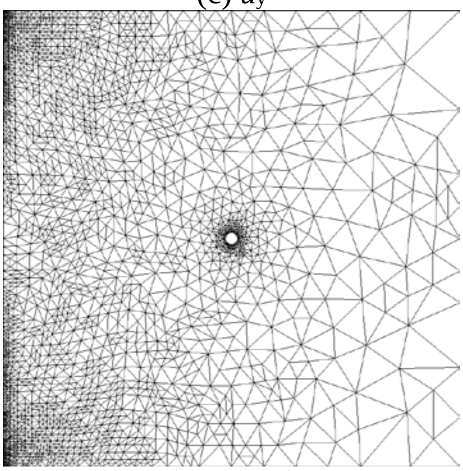
(b) u_x



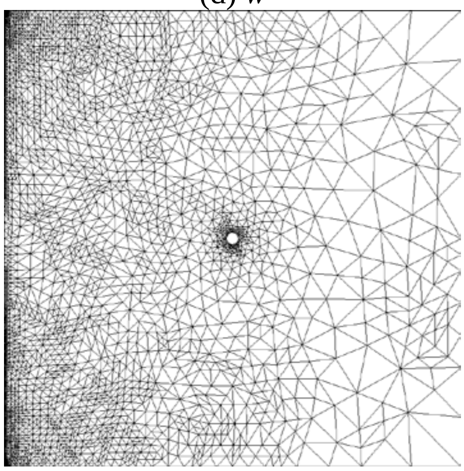
(c) u_y



(d) w



(e) 2582 nodes (reference)



(f) 4111 nodes (majorant)

Fig. 2. Example 1. Cosserat elasticity: initial mesh (a), components of the solution u and w (b-d) (displacements and rotation), the result of adaptation by the reference indicator (e), the result of adaptation by majorant-based indicator (f).

For the same problem conditions of Cosserat elasticity with the same uniform meshing, the relative error is greater than for the classical one – this effect occurs due to solving equations that are more complex from mathematical point of view.

For analyzing adaptation results, reference (target) meshes are constructed. The adaptation process takes a large amount of steps, refining only several elements on each. Elements to be refined are chosen with reference error indicators, which are based on the energy norm of the difference between solutions on coarse and fine meshes. Results for corresponding reference meshes are collected in the second block of the Table 1 and Table 3, respectively.

In the third blocks of the above-mentioned tables, the results for majorant-based adaptation process are collected. In Table 1 the functional-type error majorant from [2] is used for reliable upper error estimation. The ratio between the error majorant M and the error $|||e|||$ is used as a standard quality measure for error control. This parameter is usually called *the efficiency index* – it is denoted by I_{eff}.

The results for classical elasticity are presented in Fig. 1 with the following subplots: initial mesh, corresponding to Table 1 (a); initial mesh, corresponding to Table 2 (b); classical solution components (c-d); the final mesh for the reference indicator (e) and the final mesh for the majorant-based indicator (f). For Cosserat elasticity, the results are presented in Fig. 2 with the following subplots: initial mesh, corresponding to Table 3 (a); solution components (b-d); the final mesh for the reference indicator (e) and the final mesh for the majorant-based indicator (f). The difference between solutions of classical and Cosserat elasticity problems is moderate.

The results show that for considered parameters, geometry and loading in both cases (classical and Cosserat model) majorant-based error indicators lead to final adaptive meshes, which are similar to reference ones. The adaptation process was stopped after reaching the same error level as on uniform mesh with 71560 nodes. For classical model the number of nodes in the final adaptive mesh is 2960 and for Cosserat model – it is 4111, which is more than 10 times less. These results show that adaptive refinements save a lot of computational resources to get an approximate solution of a good quality.

It is also worth noting that adaptive meshes for different elasticity models have different structure. In first case, the node concentration regions are around the corners of clamped edge and around the hole. In the second case (Cosserat model), the node concentration region is more along the whole clamped edge.

In addition, Table 4 illustrates the behavior of error estimation for several steps with uniform mesh refinements for the simplest Arnold-Boffi-Falk approximation [65]. From these results for Cosserat elasticity we conclude that the efficiency index of estimates remains stable and overestimation of the true error is moderate and acceptable.

Table 4. Example 1. Results for the lowest order Arnold-Boffi-Falk approximation for nested meshes [5].

MESH	1	2	3	4
NODES	168	624	2400	9408
RELATIVE ERROR, %	15.8	11.1	7.3	4.0
I_{eff}	1.2	1.2	1.2	1.3

Example 2 (Γ-shaped domain). In this example the Γ-shaped domain is considered. Length of the left and upper edge is 2 m, the other edges are of length 1 m. The left edge is fully clamped and on the upper edge a loading is applied.

As for the Example 1, the results for classical elasticity model are grouped in Table 5, and for Cosserat model – in Table 6.

For this example the difference between the solutions of classical and Cosserat elasticity problems is more significant. In addition, the relative error for the solution of Cosserat elasticity problem is almost two times larger than for classical one. Nevertheless, for both elasticity models the difference in the number of nodes for final adaptive and uniform meshes with the same level of relative error is still significant.

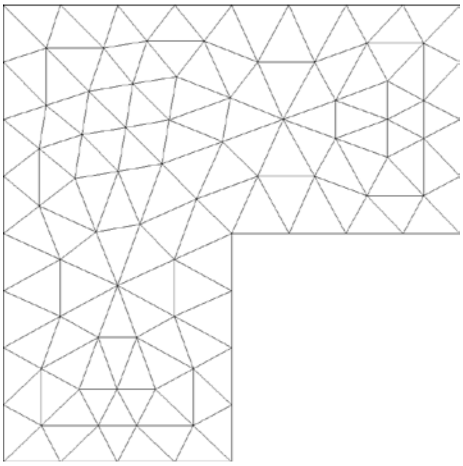
Table 5. Example 2. Classical elasticity: results for uniform and adaptive mesh refinements.

Uniform refinement						
MESH	1	2	3	4	5	6
NODES	85	305	1153	4481	17665	70145
ELEMENTS	136	544	2176	8704	34816	139264
RELATIVE ERROR, %	26.2	17.9	12.0	8.1	5.4	3.7
Reference indicator						
NODES	85	217	357	694	1591	3409
ELEMENTS	136	379	639	1278	3004	6557
RELATIVE ERROR, %	26.2	15.3	11.4	8.1	5.4	3.7
Majorant-based indicator						
NODES	85	177	532	1041	1898	3582
ELEMENTS	136	304	983	1969	3643	6942
RELATIVE ERROR, %	26.2	16.52	9.8	7.3	5.4	3.9
$I_{eff} = M/\|\|\|e\|\|\|$	1.2	1.2	1.2	1.2	1.2	1.2

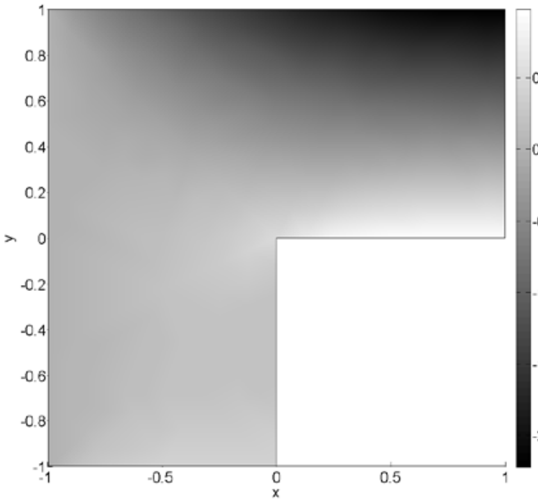
Table 6. Example 2. Cosserat elasticity: results for uniform and adaptive mesh refinements.

Uniform refinement						
MESH	1	2	3	4	5	6
NODES	85	305	1153	4481	17665	70145
ELEMENTS	136	544	2176	8704	34816	139264
RELATIVE ERROR, %	53.0	39.4	27.6	18.9	12.8	8.7
Reference indicator						
NODES	85	227	674	1640	4229	10036
ELEMENTS	136	398	1241	3073	8049	19276
RELATIVE ERROR, %	53.0	37.6	26.4	18.9	12.7	8.7
Majorant-based indicator						
NODES	85	267	943	1904	5582	15941
ELEMENTS	136	449	1680	3440	10268	29642
RELATIVE ERROR, %	53.0	37.7	26.1	20.1	12.6	7.9

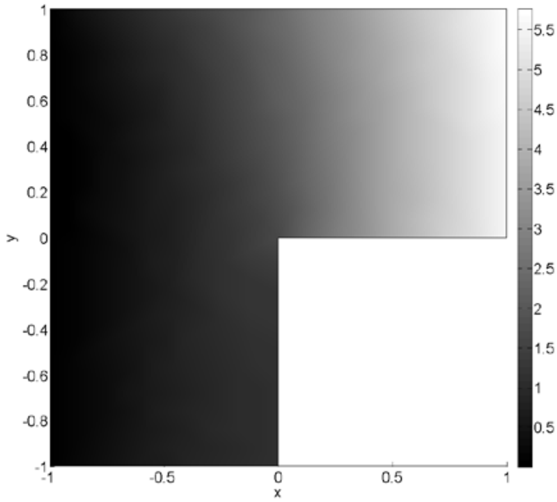
The adaptive meshes corresponding to the last columns of Table 5 and Table 6 are presented in Fig. 3 and Fig. 4. As in Example 1, the adaptive mesh structure is different for classical and Cosserat elasticity models. For the classical elasticity problem node concentration regions are around the corners of clamped edge and around the domain reentrant corner. For the Cosserat elasticity problem the node concentration region is more along the clamped edge and the domain reentrant corner.



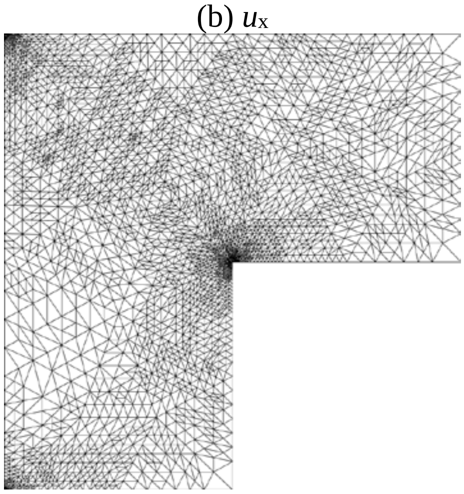
(a) 85 nodes (initial mesh)



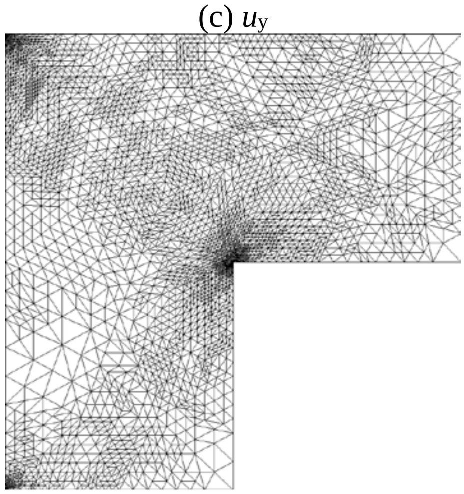
(b) u_x



(c) u_y

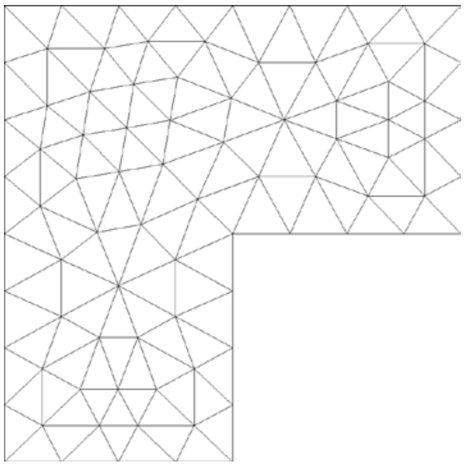


(d) 3409 nodes (reference)

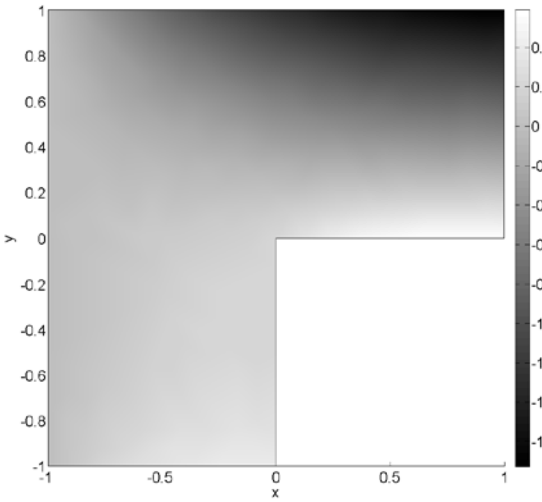


(e) 3582 nodes (majorant)

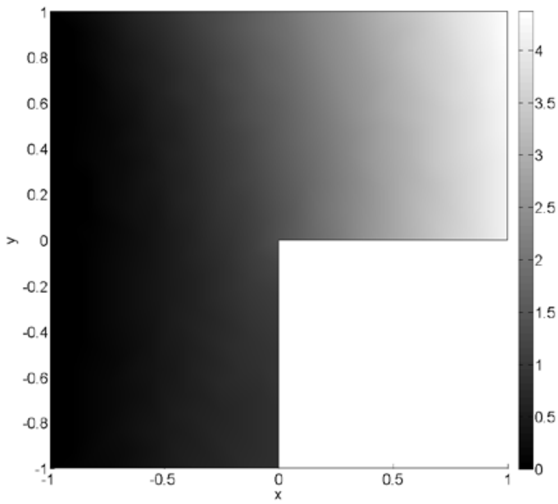
Fig. 3. Example 2. Classical elasticity: initial mesh (a), components of the solution u (b,c) (displacements), the result of adaptation by the reference indicator (d), the result of adaptation by majorant-based indicator (e).



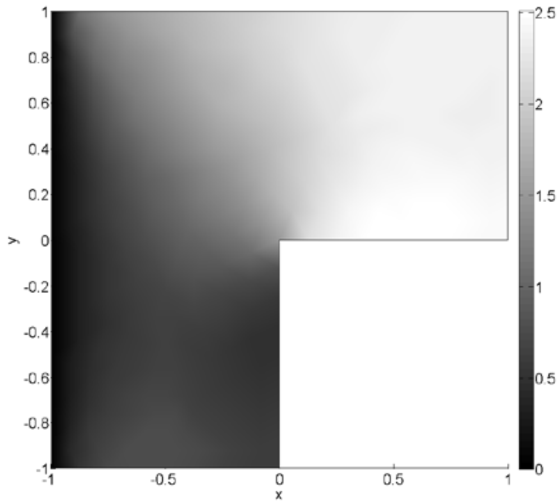
(a) 85 nodes (initial mesh)



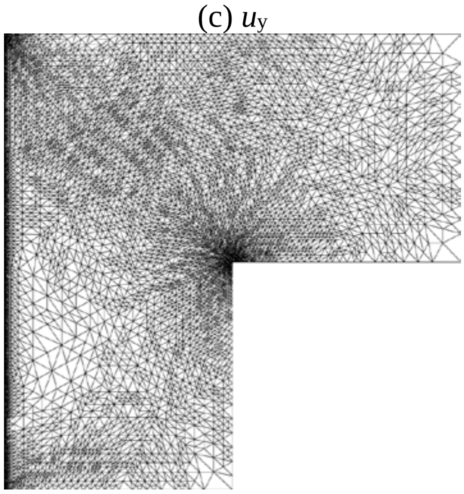
(b) u_x



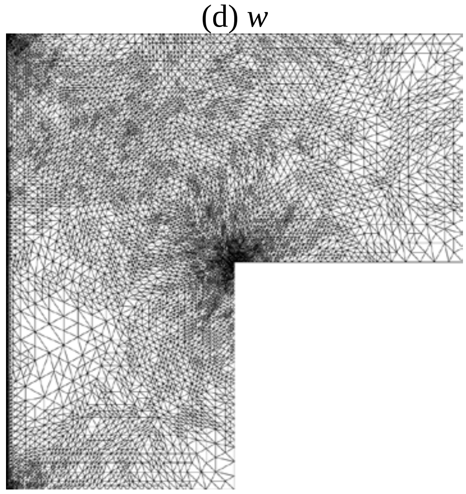
(c) u_y



(d) w



(e) 10036 nodes (reference)



(f) 15941 nodes (majorant)

Fig. 4. Example 2. Cosserat elasticity: initial mesh (a), components of the solution u and w (b-d) (displacements and rotation), the result of adaptation by the reference indicator (e), the result of adaptation by majorant-based indicator (f).

4. Conclusions

The functional approach is always reliable due to the fact that estimates are guaranteed upper bounds of errors. This property is known from the corresponding mathematical theory and it is numerically approved in the process of implementation of adaptive algorithms. As local error indicators, respective majorants provide useful information about distributions of computational errors that leads to efficient mesh adaptations and significantly saves computational resources for getting accurate approximate solutions (tens of times). For the considered classes of problems, H(div) conforming approximations as Raviart-Thomas or Arnold-Boffi-Falk yield suitable results from the viewpoint of a stability of the efficiency index and a moderate overestimation of the true error.

Acknowledgement. *Authors are supported by the Grant of the President of the Russian Federation MD-1071.2017.1.*

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