Materials Physics and Mechanics 24 (2015) 309-318 Received: October 16, 2015

MODELING OF DENSELY CRACKED SURFACES AND THE GRIFFITH ENERGY CRITERION OF FRACTURE

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Abstract. Considering an elastic homogeneous isotropic body with a periodic family of surface microcracks, it is observed and justified rigorously that an influence of the microcracks on the far-field stress-strain state of the body can be taken into account at an appropriate asymptotic precision in a certain norm by creation of an asymptotic-variational model for an elastic dummy obtained by clipping out a thin near-surface layer of the elastic material. In other words, an abatement of a solid resistance due to the surface damage is equivalent to spalling of a subsurface flake realized in the model as a regular shift of the exterior boundary along the interior normal. The asymptotic-variational model is consistent with both, the Griffith energy criterion of fracture and spectral characteristics (e.g., eigenfrequencies) of the damaged body. At the same time, the traditional modelling through so-called "wall-laws" or singularly perturbed boundary conditions of Wentzel's type leads to ill-posed spectral problems. Numerical schemes for the asymptotic-variational model in the designed regularly perturbed domain do not differ from the ones in the original elastic body with a smooth intact surface that is without microcracks that makes the proposed approach to interpret damaged surfaces efficient.

1. Problem setting

Let a planar elastic body Ω occupy a domain in the plane $\mathbb{R}^2$ enveloped by a smooth simple closed contour $\Sigma = \partial\Omega$. In a neighborhood V of the surface Σ including its d-neighborhood V_d, $d > 0$, we introduce a system of the natural curvilinear coordinates (n, s) where n is the oriented distance to Σ, $n < 0$ in Ω, and s is the arc length along Σ. By rescaling we reduce to 1 the total length of the contour Σ so that all coordinates and geometric parameters become dimensionless.

Let N be a big natural number, hence $\delta = 1/N$ is a small parameter. Dealing with a smooth profile function H on Σ, we define a family of surface microcracks

$$\gamma_m^\delta(H) = \{x = (x_1, x_2) \in V: s = \delta m, n \in [-\delta H(s), 0]\}, \qquad m = 0, \dots, N-1, \tag{1}$$

and a two-dimensional elastic body with the densely cracked surface

$$\Omega^\delta(H) = \Omega \setminus \cup_{m=0}^{N-1} \gamma_m^\delta(H). \tag{2}$$

According to definition (1), the exterior boundary Σ_N^0 of the domain (2) consists of N open arcs of the small length $\delta = 1/N$ while the origin $s = 0$, cf. (1) at $m = 0$, could be fixed rather arbitrarily. Surfaces of the crack $\gamma_m^\delta(H)$ are denoted by $\gamma_m^{\delta\pm}(H)$ so that

$$\Sigma^{\delta}(H) = \partial\Omega^{\delta}(H) = \Sigma_N^0 \cup \cup_{\pm} \cup_{m=0}^{N-1} \gamma_m^{\delta\pm}(H). \quad (3)$$

The latter component in the union (3) describes damage of the solid. Notice that the intrusion depth of the cracks (1) is ruled by the common smooth profile function H and in the case $H(s) = H_0$ they become equal in length. At the same time, the distance between neighboring microcracks is of the same order as their length, i.e., they are distributed densely.

The elastic body (2) is homogeneous isotropic with the Lamé constants $\lambda \geq 0$ and $\mu > 0$. The cracks surfaces are traction-free but the mass forces $f = (f_1, f_2)$ and the traction $g = (g_1, g_2)$ are applied in the body $\Omega^{\delta}(H)$ and at the exterior boundary Σ_N^0, being smooth in the closure $\overline{\Omega}$ and on the intact contour Σ, respectively. Then elastic fields in $\Omega^{\delta}(H)$ are described by the following boundary value problem:

$$-\partial_1 \sigma_{1k}(u^{\delta}; x) - \partial_2 \sigma_{2k}(u^{\delta}; x) = f_k(x), \qquad x \in \Omega^{\delta}(H), \quad (4)$$

$$\sigma_k^{(n)}(u^{\delta}; 0, s) = g_k(s), \qquad x \in \Sigma_N^0, \quad (5)$$

$$\sigma_k^{(n_m^{\delta\pm}(H))}(u^{\delta}; x) = 0, \qquad x \in \gamma_m^{\delta\pm}(H), \; m = 0, \dots, N-1. \quad (6)$$

Here, $k = 1,2$, $\partial_k = \partial/\partial x_k$, $n = (n_1, n_2)$ is the unit vector of the outward normal on Σ and $n_m^{\delta\pm}(H)$ stands for the same vector on the cracks surfaces $\gamma_m^{\delta\pm}(H)$. Moreover, $u^{\delta} = (u_1^{\delta}, u_2^{\delta})$ is the displacement vector and u_k^{δ} denotes its projection on the x_k-axis of the Cartesian coordinate system $x = (x_1, x_2)$ while $u^{\delta}(n, s)$ is the vector function $V \cap \Omega^{\delta}(H) \ni x \mapsto u^{\delta}(x)$ written in the local coordinates. The Cartesian components of the stress tensor are given by the linearized Cauchy formulas

$$\begin{aligned} \sigma_{11}(u^{\delta}) &= (\lambda + 2\mu)\partial_1 u_1^{\delta} + \lambda \partial_2 u_2^{\delta}, \\ \sigma_{22}(u^{\delta}) &= (\lambda + 2\mu)\partial_2 u_2^{\delta} + \lambda \partial_1 u_1^{\delta}, \\ \sigma_{12}(u^{\delta}) &= \sigma_{21}(u^{\delta}) = \mu(\partial_2 u_1^{\delta} + \partial_1 u_2^{\delta}) \end{aligned} \quad (7)$$

and the Cartesian components of the normal stress vector take the form

$$\sigma_k^{(n)}(u^{\delta}) = n_1 \sigma_{1k}(u^{\delta}) + n_2 \sigma_{2k}(u^{\delta}), \;\; k = 1,2. \quad (8)$$

The external forces f and loading g in (4) and (5) are self-balanced, that is

$$\begin{aligned} &\int_{\Omega} f_k(x)dx + \int_{\Sigma} g_k(s)ds = 0, \;\; k = 1,2, \\ &\int_{\Omega} (x_1 f_2(x) - x_2 f_1(x))dx + \int_{\Sigma} (x_1 g_2(s) - x_2 g_1(s))ds = 0. \end{aligned} \quad (9)$$

We also write formulas relating projections of the displacement vector u^{δ} and the stress tensor $\sigma(u^{\delta})$ on the axes n and s of the local system:

$$\begin{aligned} \sigma_{nn}(u^{\delta}) &= (\lambda + 2\mu)\partial_n u_n^{\delta} + \lambda J^{-1}(\partial_s u_s^{\delta} + \kappa u_n^{\delta}), \\ \sigma_{ss}(u^{\delta}) &= (\lambda + 2\mu)J^{-1}(\partial_s u_s^{\delta} + \kappa u_n^{\delta}) + \lambda \partial_n u_n^{\delta}, \\ \sigma_{ns}(u^{\delta}) &= \sigma_{sn}(u^{\delta}) = \mu\left(\partial_s u_1^{\delta} + J^{-1}(\partial_s u_n^{\delta} - \kappa u_s^{\delta})\right), \end{aligned} \quad (10)$$

where $J(n, s) = 1 + n\kappa(s)$ is the Jacobian and $\kappa(s)$ is the curvature of the contour at point $s \in \Sigma$. Furthermore, the equilibrium equations in $V_d \cap \Omega^{\delta}(H)$ take the form

$$\begin{aligned} &-\partial_n \sigma_{nn}(u^{\delta}) - J^{-1}\left(\partial_s \sigma_{ns}(u^{\delta}) + \kappa\left(\sigma_{nn}(u^{\delta}) - \sigma_{ss}(u^{\delta})\right)\right) = f_n, \\ &-\partial_n \sigma_{sn}(u^{\delta}) - J^{-1}\left(\partial_s \sigma_{ss}(u^{\delta}) + 2\kappa\sigma_{ns}(u^{\delta})\right) = f_s, \qquad s \in \Sigma, \; n \in (-d, 0), \end{aligned} \quad (11)$$

while the boundary conditions (5) and (6) turn into

$$\sigma_{nn}\big(u^{\delta};0,s\big)=g_n(s),\quad \sigma_{sn}\big(u^{\delta};0,s\big)=g_s(s),\qquad s\in\Sigma_N^0, \tag{12}$$

$$\sigma_{sn}\big(u^{\delta};n,m\delta\pm 0\big)=0,\ \ \sigma_{ss}\big(u^{\delta};n,m\delta\pm 0\big)=0,\ \ n\in(-H(m\delta),0),\ \ m=0,\dots,N-1. \tag{13}$$

We will proceed in Section 2 with describing two-term asymptotics of displacements and stresses in $\Omega^{\delta}(H)$ which involve boundary layer terms appearing as decaying solutions of the elasticity problem in the cracked semi-strip $\Pi(H)=\Pi\backslash\Gamma(H)$,

$$\Pi=\Big\{\xi=(\xi_1,\xi_2)\in\mathbb{R}^2:\xi_1<0,|\xi_2|<\tfrac{1}{2}\Big\},\quad \Gamma(H)=\{\xi:\xi_1\in[-H,0],\xi_2=0\}. \tag{14}$$

Operating with this asymptotic form we will propose and investigate some models of the damage surface (3).

2. Asymptotics of elastic fields

Using an asymptotic procedure developed in [1, 2], we search for a solution of problem (4)-(6) in the asymptotic form

$$u^{\delta}(x)=u^{0}(x)+\delta\left(u'(x)+\chi(x)w(\delta^{-1}n,\delta^{-1}s;s)\right)+\tilde{u}^{\delta}(x), \tag{15}$$

where the main term u^0 is a solution of the limit ($\delta=0$) problem

$$-\partial_1\sigma_{1k}(u^0;x)-\partial_2\sigma_{2k}(u^0;x)=f_k(x),\qquad x\in\Omega,\quad k=1,2, \tag{16}$$

$$\sigma_{nn}(u^0;0,s)=g_n(s),\quad \sigma_{sn}(u^0;0,s)=g_s(s),\qquad s\in\Sigma. \tag{17}$$

Owing to the orthogonality conditions (9), this problem has a solution u^0 in the Sobolev space $H^2(\Omega)^2$ (the latter superscript 2 indicates the number of components in vector functions).

Let us describe correction terms in (15). The boundary layer term w is written in the stretched curvilinear coordinates

$$\xi=(\xi_1,\xi_2),\qquad\qquad \xi_1=\delta^{-1}n,\ \ \xi_2=\delta^{-1}s, \tag{18}$$

and is localized in the vicinity of Σ by means of a smooth cut-off function χ such that $\chi(x)=1$ for $|n|<d/3$ and $\chi(x)=0$ for $|n|>2d/3$. Furthermore, to satisfy the boundary conditions (6) on the crack surfaces, we impose the representation

$$w(\xi;s)=-\sigma_{ss}(u^0;0,s)W(\xi;H(s)), \tag{19}$$

where $W(\cdot\,;H(s))$ is a solution of the following problem in the elastic semi-strip Π with the surface crack $\Gamma(H)$ of length $H>0$, see (14),

$$-\partial_1\sigma_{1k}(W;\xi)-\partial_2\sigma_{2k}(W;\xi)=0,\qquad \xi\in\Pi(H)=\Pi\backslash\Gamma(H), \tag{20}$$

$$\sigma_{11}(W;0,\xi_2)=0,\ \ \sigma_{21}(W;0,\xi_2)=0,\qquad |\xi_2|<1/2, \tag{21}$$

$$\sigma_{22}(W;\xi_1,\pm 0)=\mp 1,\ \ \sigma_{12}(W;\xi_1,\pm 0)=0,\qquad \xi_1\in(-H,0), \tag{22}$$

$$W_k\left(\xi_1,+\tfrac{1}{2}\right)=W_k\left(\xi_1,-\tfrac{1}{2}\right),\ \sigma_{1k}\left(W;\xi_1,+\tfrac{1}{2}\right)=\sigma_{1k}\left(W;\xi_1,-\tfrac{1}{2}\right),\ k=1,2,\ \xi_1<0. \tag{23}$$

After the strong dilation, the coordinates (18) here are assumed to compose a Cartesian system so that $\partial_k=\partial/\partial\xi_k$ in (20) and the stresses $\sigma_{11}(W)=\delta\sigma_{nn}(W)$, $\sigma_{22}(W)=\delta\sigma_{ss}(W)$, $\sigma_{12}(W)=\delta\sigma_{ns}(W)$ are calculated according to formulas (7), instead of (10), with the corresponding replacement $x\longmapsto\xi$; notice that the factor δ results from the present usage of the stretched coordinates (18). The very reason for such the interpretation of coordinates (18) is based on the following observations. First, the Jacobian in V_d becomes $J(n,s)=1+\delta\xi_1\kappa(s)=1+O(\delta)$ and, therefore, we can omit the coefficient J^{-1} in all formulas (10) and

(11). Second, the coordinate dilation $(n,s)\mapsto(\xi_1,\xi_2)$ brings the big factor δ^{-1} on derivatives, namely $\partial/\partial\xi_1=\delta^{-1}\partial_n, \partial/\partial\xi_2=\delta^{-1}\partial_s$ so that in (10) and (11) terms with the curvature $\kappa(s)$ are neglectable in the limit as well.

The punctured end $\{\xi : \xi_1 = 0, |\xi_2| < 1/2\}$ of the cracked semi-strip $\Pi(H)$ is traction-free but crack's surfaces are under unit normal stretching loading, see the boundary conditions (21) and (22), respectively. The periodicity conditions (23) close the problem and, since any rigid motion translation satisfies the homogeneous equations (20)-(23), the problem has a unique solution $W(\cdot\,;H)\in H^1(\Pi(H))^2$ with the exponential decay as $\xi_1\to-\infty$. The latter assures the desired localization effect for the boundary layer term. In view of square-root singularities of the stresses $\sigma_{jk}(W)$ at the crack tip the solution does not belong to $H^2(\Pi(H))^2$, however it is smooth everywhere in $\overline{\Pi(H)}$ with exception of three corner points $(-H,0)$ and $(\pm0,0)$. Notice that the periodicity conditions (23) identify the corners with the tops $(0,\pm1/2)$ and they do not produce singularities of elastic fields.

In the sequel we will need the following integral characteristics of the cracked strip $\Pi(H)$:

$$0<E_H:=E_H(W)=\frac{1}{2}\int_{\Pi(H)}\left(2\mu\left|\frac{\partial W_1}{\partial x_1}(\xi;H)\right|^2+2\mu\left|\frac{\partial W_2}{\partial x_2}(\xi;H)\right|^2+\right.$$

$$\left.+\mu\left|\frac{\partial W_2}{\partial x_1}(\xi;H)+\frac{\partial W_1}{\partial x_2}(\xi;H)\right|^2+\lambda\left|\frac{\partial W_1}{\partial x_1}(\xi;H)+\frac{\partial W_1}{\partial x_1}(\xi;H)\right|^2\right)d\xi. \tag{24}$$

The right-hand side of (24) is nothing but the elastic energy kept in the solid $\Pi(H)$. Recall that the inclusion $W(\cdot;H)\in H^1(\Pi(H))^2$ makes the integral in (24) convergent.

Endowing the second boundary layer term $\delta^2 w''(\xi;s)$ with the decay property, i.e., satisfying certain solvability conditions in $\Pi(H)$, a long array of computations performed in [2] leads to the inhomogeneous boundary conditions

$$\sigma_{nn}(u';0,s)=\kappa(s)\alpha(s)\sigma_{ss}(u^0;0,s),\quad \sigma_{ns}(u';0,s)=-\partial_s\big(\alpha(s)\sigma_{ss}(u^0;0,s)\big),\quad s\in\Sigma, \tag{25}$$

for the differential equations

$$-\partial_1\,\sigma_{1k}(u';x)-\partial_2\,\sigma_{2k}(u';x)=0,\qquad x\in\Omega,\qquad k=1,2. \tag{26}$$

The function coefficient $\alpha(s)$ in (25) is as follows:

$$\alpha(s)=ME_{H(s)},\qquad M=\frac{4\mu(\lambda+\mu)}{\lambda+2\mu}. \tag{27}$$

We emphasize that both, the coefficient (27) and the curvature $\kappa(s)$ are dimensionless. We also write down the relations

$$\sigma_{ss}(u^0;0,s)=M\varepsilon_{ss}(u^0;0,s)-Lg_s(s),\quad L=\frac{\lambda}{\lambda+2\mu}, \tag{28}$$

which are inherited from (10), (17) and involve the tangential strain $\varepsilon_{ss}(u^0;0,s)=J^{-1}\big(\partial_s u_s^\delta+\kappa u_n^\delta\big)$ at the contour Σ.

Clearly, the loading applied to the boundary Σ in (25) is self-balanced, cf. (9). Hence, the solution u' exists but is defined up to a rigid motion. The orthogonality conditions

$$\int_{\Omega^\delta(H)}u_k^\delta(x)dx=0,\ \ k=1,2,\qquad \int_{\Omega^\delta(H)}\left(x_1u_2^\delta(x)-x_2u_1^\delta(x)\right)dx=0 \tag{29}$$

for the solution of the original problem (4)-(6) also apply in a clear way to the solutions u^0 and u' of problems (16), (17) and (26), (25). These determine uniquely all ingredients of the asymptotic ansatz (15). An estimate of the remainder $\tilde u^\delta$ derived in [2] and based on the weighted Korn inequality [3] in domains with rough boundaries, reads:

$$\|\nabla\,\tilde{u}^{\delta};\ L^2(\Omega^{\delta}(H))\| + \|\,\tilde{u}^{\delta};\ L^2(\Omega^{\delta}(H))\| \le c\delta^{3/2}. \tag{30}$$

Here, $L^2(\Omega^{\delta}(H))^2$ is the Lebesgue space so that the left-hand side of (30) exhibits the standard Sobolev norm in the space $H^1(\Omega^{\delta}(H))^2$.

3. Modeling a damaged surface

We, first of all, observe that, owing to the exponential decay of the vector function W in (19), the Lebesgue norm of the boundary layer term $\delta\chi w$ is $O(\,\delta^{3/2})$ while the additional exponent ½ of δ comes in from the integral $\left(\int_{-d}^{0} e^{\tau n/\delta} dn\right)^{1/2} = O(\,\delta^{1/2})$ which appears with some $\tau > 0$ just due to the above-mentioned decay property. Thus, the estimate (30) without this term can be converted into

$$\|\,u^{\delta} - v^{\delta}\,;\ L^2(\Omega^{\delta}(H))\| \le c\delta^{3/2}, \tag{31}$$

where the sum v^{δ} of two regular terms from (15) figure in, namely

$$v^{\delta}(x) = u^0(x) + \delta u'(x). \tag{32}$$

Notice that in view of the stretched coordinates (15) in the boundary layer term $w(\xi; s)$, absent in (32), the norm of gradient of the difference $u^{\delta} - v^{\delta}$ has order $\delta^{1/2}$ and therefore the Sobolev norm cannot substitute for the Lebesgue one in (31). In other words, the simplistic estimate (31) is much weaker than the original estimate (30). However, getting rid of the boundary layer term in the sum (32) makes the asymptotic approximation (32) much simpler in comparison with the full asymptotic expansion (15). This simplification can be used to create primary asymptotic-variational models which do not care about localized effects, i.e., the near-field, but about the far-field approximation only. Let us first discuss a traditional model involving boundary conditions of Wentzel's type, also known in the mathematical literature as "wall-laws".

We compose the boundary value problem in the intact that is without the mocrocracks (1), planar solid Ω

$$-\,\partial_1\,\sigma_{1k}(\hat{v}^{\delta}; x) - \,\partial_2\,\sigma_{2k}(\hat{v}^{\delta}; x) = f(x), \qquad x\,\in\Omega, \quad k = 1,2\,, \tag{33}$$

$$\sigma_{nn}(\hat{v}^{\delta}; 0, s) = \hat{g}_n^{\delta}(s) + \delta\kappa(s)\hat{\alpha}(s)\varepsilon_{ss}(\hat{v}^{\delta}; 0, s),$$

$$\sigma_{sn}(\hat{v}^{\delta}; 0, s) = \hat{g}_s^{\delta}(s) - \delta\partial_s\left(\hat{\alpha}(s)\varepsilon_{ss}(\hat{v}^{\delta}; 0, s)\right), \quad s\in\Sigma, \tag{34}$$

where

$$\hat{\alpha}(s) = M\alpha(s) = M^2 E_{H(s)} > 0 \tag{35}$$

and

$$\hat{g}_n^{\delta}(s) = g_n(s) - \delta L\kappa(s)\hat{\alpha}(s)g_s(s),\ \hat{g}_s^{\delta}(s) = g_s(s) + \delta L\partial_s(\hat{\alpha}(s)g_s(s)). \tag{36}$$

The boundary conditions (34) were constructed in two steps. First, we multiply (25) with δ and add the result to (17). Second, we use the notation (32) together with formula (28) and, furthermore, replace $v^{\delta}(x)$ and $\delta u^0(x)$ by $\hat{v}^{\delta}(x)$ and $\delta\hat{v}^{\delta}(x)$ respectively.

Since one of the boundary conditions (35) involves a second-order differential operator on the boundary with the small coefficient δ, the problem is to be regarded as a singularly perturbed problem of Wentzel's type. At the same time, the sum (32) leaves a small, of order δ^2, discrepancy and the asymptotic precision estimate

$$\|\nabla(\hat{v}^{\delta} - v^{\delta})\,;\ L^2(\Omega)\| + \|\hat{v}^{\delta} - v^{\delta}\,;\ L^2(\Omega)\| \le c\delta^2 \tag{37}$$

can be easily verified under a surplus smoothness assumption on the datum $\hat{g}^\delta$, see, e.g., [3, 2]. Nevertheless, the variational formulation of the problem (33), (34), namely the integral identity with arbitrary infinitely smooth test vector function $\hat{v}^\delta \in C^\infty(\overline{\Omega})^2$

$$\sum_{j,k=1,2} \int_\Omega \sigma_{jk}(\hat{v}^\delta;x)\varepsilon_{jk}(\hat{w}^\delta;x)dx - \delta \int_\Sigma \hat{\alpha}(s)\varepsilon_{ss}(\hat{v}^\delta;0,s)\varepsilon_{ss}(\hat{w}^\delta;0,s)ds =$$
$$= \sum_{k=1,2} \int_\Omega f_k(x)\, \hat{w}_k^\delta(x)dx + \sum_{k=1,2} \int_\Sigma \hat{g}_k^\delta(0,s)\, \hat{w}_k^\delta(0,s)ds \tag{38}$$

has an obvious lack: the quadratic form on the left of (38) is not positive definite due to integral over Σ with minus sign. Although the small factor δ of this integral makes the problem solvable under above-mentioned smoothness assumption on the datum $\hat{g}^\delta$, many numerical schemes cannot be applied directly to the variational problem (38) with the indefinite quadratic form. Moreover, the spectral (eigenvalue) problem considered in Section 4 becomes ill-posed and, based on an idea in [4], we will improve in Section 5 the formulation of the combined problem.

4. Regular abatement of a loose surface

Let D be a smooth positive profile function in Σ. We cut off a near-surface layer of depth $\delta D(s)$ from the original body Ω, namely

$$\Omega_D^\delta = (\Omega \backslash V) \cup \{x \in \Omega \cap V : n < -\delta D(s)\} \subset \Omega, \tag{39}$$

where $\delta > 0$ is again a small parameter. In the domain (39) we consider the boundary value problem

$$-\partial_1 \sigma_{1k}(\breve{u}^\delta;x) - \partial_2 \sigma_{2k}(\breve{u}^\delta;x) = f_k(x), \qquad x \in \Omega_D^\delta, \quad k = 1,2, \tag{40}$$

$$\sigma_k^{(n_D^\delta)}(\breve{u}^\delta; -\delta D(s), s) = g_k(s) + \delta g_k'(s) + \tilde{g}_k^\delta(s), \qquad s \in \Sigma, \quad k = 1,2. \tag{41}$$

Here, n_D^δ is the outward normal unit vector at the boundary $\Sigma_D^\delta = \partial\Omega_D^\delta$,

$$(n_D^\delta)_n(s) = N(s;\delta) = 1 + O(\delta^2),$$
$$(n_D^\delta)_s(s) = N(s;\delta)\,(1 - \delta D(s)\kappa(s))^{-2}\delta\partial_s D(s) = \delta\partial_s D(s) + O(\delta^2), \tag{42}$$
$$N(s;\delta) = (1 + \delta^2\,(1 - \delta D(s)\kappa(s))^{-2}|\partial_s D(s)|^2)^{-1/2}.$$

Higher-order terms were introduced on the right-hand side of (41) in order to support the compatibility conditions (9) in the perturbed domain (39). The main correction term $g_k'(s)$ in (41) will be computed but the remainder $\tilde{g}_k^\delta(s) = O(\delta^2)$ becomes of no importance in our calculations.

We accept the standard and simplest ansatz

$$\breve{u}^\delta(x) = u^0(x) + \delta\, \breve{u}'(x) + \tilde{\breve{u}}^\delta(x), \tag{43}$$

where u^0 is the above-used solution of the limit problem (16), (17). The next term satisfies the equilibrium equations (26). To close a problem for $\breve{u}'$, we need to transfer the boundary conditions (41) from the perturbed contour Σ_D^δ onto the reference contour Σ. To this end, we first of all write relations for projections of vectors on the n_D^δ- and s_D^δ-axes of the local coordinates system near the arc Σ_D^δ:

$$\sigma_{n_D^\delta n_D^\delta}(\breve{u}^\delta;x) = \sigma_{nn}(\breve{u}^\delta;x) + 2\delta\partial_s D(s)\sigma_{sn}(\breve{u}^\delta;x) + O(\delta^2),$$
$$\sigma_{n_D^\delta s_D^\delta}(\breve{u}^\delta;x) = \sigma_{ns}(\breve{u}^\delta;x) - \delta\partial_s D(s)\left(\sigma_{nn}(\breve{u}^\delta;x) - \sigma_{ss}(\breve{u}^\delta;x)\right) + O(\delta^2),$$
$$g_{n_D^\delta}^\delta(s) = g_n(s) + \delta\left(g_n'(s) + \partial_s D(s) g_s(s)\right) + O(\delta^2), \tag{44}$$

$$g^{\delta}_{s^{\delta}_{D}}(s) = g_s(s) + \delta\big(g'_s(s) - \partial_s D(s) g_n(s)\big) + O(\delta^2),$$

where $s^{\delta}_{D} = \left(-\left(n^{\delta}_{D}\right)_s, \left(n^{\delta}_{D}\right)_n\right)$ stands for the tangential unit vector at the contour Σ, cf., formulas (42) for the normal vector. Relation (44) inherited from (10) and (42), are inserted into the boundary conditions (41) and, after applying the Taylor formula in the variable n, we collect coefficients of δ and obtain

$$\sigma_{nn}(\breve{u}'; 0, s) = D(s)\partial_n \sigma_{nn}(u^0; 0, s) - 2\partial_s D(s)\sigma_{ns}(u^0; 0, s) + g'_n(s) + \partial_s D(s) g_s(s),$$

$$\sigma_{ns}(\breve{u}'; 0, s) = D(s)\partial_n \sigma_{ns}(u^0; 0, s) - 2\partial_s D(s)\big(\sigma_{nn}(u^0; 0, s) - \sigma_{ss}(u^0; 0, s)\big) +$$

$$+ g'_n(s) + \partial_s D(s) g_s(s). \tag{45}$$

Setting $n = 0$ in the differential equations (16) rewritten in the form (11) yields

$$\partial_n \sigma_{nn}(u^0; 0, s) = -f_n(0, s) - \partial_s \sigma_{ns}(u^0; 0, s) - \kappa(s)\big(\sigma_{nn}(u^0; 0, s) - \sigma_{ss}(u^0; 0, s)\big),$$

$$\partial_n \sigma_{ns}(u^0; 0, s) = -f_s(0, s) - \partial_s \sigma_{ss}(u^0; 0, s) - 2\kappa(s)\sigma_{ns}(u^0; 0, s). \tag{46}$$

Taking (46) into account, we fix

$$g'_n(s) = D(s)\big(f_n(0, s) + \partial_s g_s(s) + \kappa(s) g_n(s)\big) - \partial_s D(s) g_s(s),$$

$$g'_s(s) = D(s)\big(f_s(0, s) - 2\kappa(s) g_s(s)\big) - \partial_s D(s)\big(g_n(s) - g_s(s)\big), \tag{47}$$

and, therefore, the equations (45) become

$$\sigma_{nn}(\breve{u}'; 0, s) = -\kappa(s) D(s)\sigma_{ss}(u^0; 0, s), \sigma_{ns}(\breve{u}'; 0, s) = \partial_s\left(D(s)\,\sigma_{ss}(u^0; 0, s)\right), \ s \in \Sigma. \tag{48}$$

Remark. In the case $g = 0$ the necessity to introduce the addendum $\delta g'(s) = \delta D(s) f(0, s)$, see (47), onto the right-hand side of (44) is explained by the asymptotic representation

$$\int_{\Omega^{\delta}_{D}} f(x)dx = \int_{\Omega} f(x)dx - \int_{\Omega\setminus\Omega^{\delta}_{D}} f(x)dx = \int_{\Omega} f(x)dx - \delta\int_{\Sigma} D(s) f(0, s)ds +$$

$$+ O(\delta^2).$$ ■

Comparing (25) and (48), we conclude that in the case

$$D(s) = \alpha(s), \quad s \in \Sigma, \tag{49}$$

the main correction terms $\breve{u}'$ in the ansatz (43) and u' in the ansatz (15) coincide with each other, that is $\breve{u}' = u'$. Moreover, the evident estimate of the remainder $\tilde{\breve{u}}^{\delta}(x)$ in (43)

$$\left\| \breve{u}^{\delta} - u^0 - \delta\, \breve{u}' ;\ L^2\big(\Omega^{\delta}_{D}\big)\right\| \le c\delta^{3/2}, \tag{50}$$

cf., [4; Section 7.6.5], combined with (31) and (32), demonstrates that

$$\left\| u^{\delta} - \breve{u}^{\delta} ;\ L^2\big(\Omega^{\delta}_{D}\big)\right\| \le c\delta^{3/2}. \tag{51}$$

In other words, the solution of the model problem (40), (41) in the domain Ω^{δ}_{D} with a regularly perturbed but smooth boundary Γ^{δ}_{D} gives a good approximation, although in the weaker norm of $L^2\big(\Omega^{\delta}_{D}\big)^2$, for the solution of the original problem (4)-(6) in the domain $\Omega^{\delta}(H)$ with singularly perturbed piecewise smooth boundary $\Gamma^{\delta}(H)$, namely the damaged surface (3). Beyond any doubt solving numerically a problem in a smooth domain is much simpler than in a domain with rapidly oscillating boundary and plenty of corner points. The parameter $D(s)$, that is depth of boundary's shift along the inward normal, in the regular perturbation (39) of the reference domain Ω is computed, according to (49), (27), (24), through the solution $W(\xi; H(s))$ of a standard elasticity boundary value problem in the cracked semi-strip (14) and the energy functional (24) can be tabulated. Moreover, in Section 6 it will be observed that the integral characteristics $E_{H(s)}$ in use is related to stress intensity factors at the tip of the crack $\Gamma(H)$ which can be found out in [5] and other handbooks on fracture mechanics.

5. The spectral problem

The weak estimate of the asymptotic correlation between solutions of the perturbed problems (4)-(6) and (40), (41) is useful in many aspects. Let us outline a result in [1, 7] about the eigenvalue elasticity problem consisting of the differential equations

$$-\partial_1 \sigma_{1k}(u^\delta;x) - \partial_2 \sigma_{2k}(u^\delta;x) = \omega_\delta^2 u^\delta(x), \qquad x \in \Omega^\delta(H), \quad k=1,2, \tag{52}$$

with the boundary conditions (6) and

$$\sigma_k^{(n)}(u^\delta;0,s) = 0, \qquad x \in \Sigma_N^0, \quad k=1,2. \tag{53}$$

Here, $\omega_\delta > 0$ and $u^\delta(x)$ respectively are the eigenfrequency and the corresponding eigenmode in the solid $\Omega^\delta(H)$ with the damaged surface. The problem (52), (53), (6) gets the monotone unbounded eigenvalue sequence

$$0 = \omega_\delta^{(1)} = \omega_\delta^{(2)} = \omega_\delta^{(3)} < \omega_\delta^{(4)} \le \omega_\delta^{(5)} \le \cdots \le \omega_\delta^{(p)} \le \ldots, \tag{54}$$

where three null frequencies, of course, are generated by rigid motion.

Since, formally $f(x) = \omega_\delta^2 u^\delta(x)$, compare (52) and (4), the relations (47) and (45) demonstrate that the problem (40), (41) reduces to

$$-\partial_1 \sigma_{1k}(\breve{u}^\delta;x) - \partial_2 \sigma_{2k}(\breve{u}^\delta;x) = \breve{\omega}_\delta^2 \breve{u}^\delta(x), \qquad x \in \Omega_D^\delta, \; k=1,2, \tag{55}$$

$$\sigma_{n_D^\delta n_D^\delta}(\breve{u}^\delta; -\delta D(s), s) = \delta D(s)\, \breve{\omega}_\delta^2\, \breve{u}_{n_D^\delta}^\delta(0,s),$$

$$\sigma_{n_D^\delta s_D^\delta}(\breve{u}^\delta; -\delta D(s), s) = \delta D(s)\, \breve{\omega}_\delta^2\, \breve{u}_{s_D^\delta}^\delta(0,s), \qquad s \in \Sigma_D^\delta. \tag{56}$$

Although the spectral parameter $\breve{\omega}_\delta^2$ has entered the right-hand side of (56) and made these boundary conditions of Steklov's type, the spectral problem (55), (56) is well-posed and possesses the eigenvalue sequence of the same type as in (54),

$$0 = \breve{\omega}_\delta^{(1)} = \breve{\omega}_\delta^{(2)} = \breve{\omega}_\delta^{(3)} < \breve{\omega}_\delta^{(4)} \le \breve{\omega}_\delta^{(5)} \le \cdots \le \breve{\omega}_\delta^{(p)} \le \cdots. \tag{57}$$

The variational formulation of the problem (55), (56) reads as the integral identity

$$\sum_{j,k=1,2} \int_{\Omega_D^\delta} \sigma_{jk}(\breve{u}^\delta;x)\varepsilon_{jk}(\breve{w}^\delta;x)dx =$$

$$= \breve{\omega}_\delta^2 \sum_{k=1,2}\left(\int_{\Omega_D^\delta} \breve{u}_k^\delta(x)\breve{w}_k^\delta(x)dx + \delta \int_{\Sigma_D^\delta} D(s)\; \breve{u}_k^\delta(0,s)\, \breve{w}_k^\delta(0,s)ds\right) \tag{58}$$

with arbitrary test vector function $\breve{w}_k^\delta \in H^1(\Omega_D^\delta)^2$. Clearly, a rigid motion satisfies this problem with $\breve{\omega}_\delta = 0$ and, indeed, the first three eigenfrequencies in (57) are null as in (54).

As was proved in [5], entries of the sequences (54) and (57) are in the relationship

$$\left|\breve{\omega}_\delta^{(p)} - \omega_\delta^{(p)}\right| \le C_p\, \delta^{\frac{3}{2}}, \quad p = 4,5,6,\ldots, \text{ for } \delta \in (0,\delta_p). \tag{59}$$

Here, δ_p and C_p stand for some positive values depending on the serial number p of eigenvalue. A similar relationship occurs for the eigenmodes $u_{(p)}^\delta$ and $\breve{u}_{(p)}^\delta$ in the Lebesgue space $L^2(\Omega_D^\delta)^2$, compare with (51).

It is worth to emphasize again that the eigenvalue sequence for the model with the Wentzel boundary conditions, namely the homogeneous equations (34), contains in addition to the null eigenvalue of multiplicity 3 both, positive and negative eigenvalues. Negative eigenvalues give rise to pure imaginary eigenfrequencies that have no physical sense. The model (55), (56) in the regularly perturbed domain Ω_D^δ is deprived of this flaw. Moreover, numerical schemes to solve the variational spectral problem (58) in the smooth domain Ω_D^δ require for alike computer expenses as for the spectral elasticity problem in the original unperturbed body Ω that is without any damage while it is incomparably more complicated to solve the original spectral problem in the domain damaged with the family (1) of surface microcracks. It should be mentioned that each of eigenvalues in (54) can be regarded as a functional on the corresponding eigenmode. In this way an error estimate in the Lebesgue norm is sufficient to conclude with the inequality (59).

6. The Griffith energy criterion of fracture

First of all, we mention that the positive function $(0,+\infty) \ni H \to E_H$ from (24) is strictly monotone increasing. To make this conclusion, we consider the minimization of the potential energy functional for the problem (20)-(23)

$$\min\left\{E_H(U) - \sum_\pm \int_{-H}^0 U_2(\xi_1, \pm 0) d\,\xi_1 : \ U \in \mathcal{B}_H \coloneqq H^1(\Pi(H))^2 \right\} \tag{60}$$

and observe that the solution $W(\xi; H)$ of this problem is just a unique minimizer for (60) while, according to the Green formula in $\Pi(H)$, we have

$$E_H(W) - \sum_\pm \int_{-H}^0 W_2(\xi_1, \pm 0) d\,\xi_1 = -E_H(W) =: -E_H. \tag{61}$$

When the crack $\Gamma(H)$ grows, the function space where test vector functions are taken from, enlarges, that is $\mathcal{B}_{H+\Delta H} \supset \mathcal{B}_H$ for $\Delta H > 0$ because the extended crack allows for jumps of elastic fields between its surfaces. Thus, the change $H \mapsto H + \Delta H$ makes the minimum in (60) smaller but, in view of formula (61) with minus sign on the right, the desired relation $E_{H+\Delta H} > E_H$ holds true. It is worth here to mention a result in [8], namely application of the unit loading on surfaces of the incremental cut $\Gamma(H + \Delta H)\backslash\Gamma(H)$ of the small length ΔH inputs into the functional (60) an infinitesimal value $O\left(|\Delta H|^{3/2}\right)$ of a higher order.

Notice that the classical Griffith formula allows to express the energy increment $E_{H+\Delta H}(W) - E_H(W)$ through the stress intensity factor at the tip of the crack $\Gamma(H)$.

We assume that the profile function $H(s)$ in (1) gets a small positive increment $\Delta H(s)$ and therefore all microcracks have grown a bit. Putting $\mathcal{H}(s) = H(s) + \Delta H(s)$, we denote by $\mathcal{U}^\delta(x)$ the solution of the problem (4)-(6) in the perturbed domain $\Omega^\delta(\mathcal{H})$ and set

$$\mathcal{A}(s) = ME_{\mathcal{H}(s)}. \tag{62}$$

Furthermore, $\breve{\mathcal{U}}^\delta(x)$ stands for the solution of the problem (40), (41) in the domain $\Omega_\mathcal{D}^\delta$ given by formula (39) with the new profile function $\mathcal{D}(s) = \mathcal{A}(s)$, see (62) and (49).

The total increment $\Delta\mathcal{L}$ of the cracks length can be calculated as follows:

$$\Delta\mathcal{L} = \sum_{m=0}^{N-1} \delta\Delta H(m\delta) = \delta\int_\Sigma \Delta H(s)\, ds + O(\delta^2) \tag{63}$$

and thus, according to the Griffith formula, the increment $\Delta\mathcal{S}$ of the surface energy $\mathcal{S}$ becomes equal to

$$\Delta\mathcal{S} = 2\mathcal{s}\delta\int_\Sigma \Delta H(s)\, ds + O(\delta^2). \tag{64}$$

Here, $\mathcal{s} > 0$ is the density of the surface energy. Notice that, recalling the rescaling made in Section 1, the relation (63) can be rewritten in the form

$$\Delta\mathcal{S} = 2\mathcal{s}\,\Delta\mathcal{V} + O(\delta^2) = \mathcal{v}\,\Delta\mathcal{V} + O(\delta^2), \tag{65}$$

where $\Delta\mathcal{V} > 0$ is the decrement of the volume of the elastic dummy body (39) due to the growth of the microcracks. In other words, $\Delta\mathcal{V}$ is the difference of the areas of Ω_D^δ and $\Omega_\mathcal{D}^\delta$. At the same time, the factor $\mathscr{v} = 2\mathscr{s}$ on the right-hand side of (65) can be considered as the density of the volume energy caused by spilling elastic material off. In other words, $\mathscr{v}$ is the aggregate crack surface per volume unit.

To simplify intermediate calculations, we suppose for a while that the traction g on the exterior surface Σ_N^0 of the damaged body $\Omega^\delta(H)$ is absent and the volume forces vanish in the d-neighborhood V_d of Σ_N^0. Hence, the right-hand sides in the boundary conditions (5) and (41) vanish. Then, similarly to (61), the potential energies of the bodies $\Omega^\delta(\mathcal{H})$ and $\Omega_{\mathcal{D}}^\delta$ are calculated as follows:

$$\mathcal{P}^\delta(\mathcal{H}) = -\frac{1}{2}\sum_{k=1,2}\int_\Omega \mathcal{U}_k^\delta(x) f_k(x)dx,\ \mathcal{P}_{\mathcal{D}}^\delta = -\frac{1}{2}\sum_{k=1,2}\int_\Omega \breve{\mathcal{U}}_k^\delta(x) f_k(x)dx. \qquad (66)$$

We emphasize that the integration domain Ω is the same in both the integrals because $f_k = 0$ in $\Omega^\delta(\mathcal{H}) \cap V_d$ as well as in $\Omega_{\mathcal{D}}^\delta \cap V_d$. Moreover, using the estimate (51) relating the displacement fields $\mathcal{U}^\delta(x)$ in the cracked body $\Omega^\delta(\mathcal{H})$ and $\breve{\mathcal{U}}^\delta(x)$ in the elastic dummy $\Omega_{\mathcal{D}}^\delta$ we derive that

$$\left|\mathcal{P}^\delta(\mathcal{H}) - \mathcal{P}_{\mathcal{D}}^\delta\right| \le c\delta^{\frac{3}{2}}. \qquad (67)$$

This error estimate again gets a supplementary smallness order and remains valid even in the general case of non-nil traction g. The formula (67) applied for $\mathcal{H}(s) = H(s) + \Delta H(s)$ and $\mathcal{H}(s) = H(s)$ leads to the relationship

$$\left|\Delta\mathcal{P}^\delta(\mathcal{H}) - \Delta\mathcal{P}_{\mathcal{D}}^\delta\right| \le c\delta^{\frac{3}{2}} \qquad (68)$$

between increments of potential energy in the damaged body $\Omega^\delta(\mathcal{H})$ and in the dummy $\Omega_{\mathcal{D}}^\delta$, that is

$$\Delta\mathcal{P}^\delta(H) = \mathcal{P}^\delta(H + \Delta H) - \mathcal{P}^\delta(H), \qquad \Delta\mathcal{P}_D^\delta = \mathcal{P}_{D+\Delta D}^\delta - \mathcal{P}_D^\delta$$

with a clear meaning of the increment ΔD of the profile function in (39).

As a result, the inequality (68) together with relations (64) and (65) helps us to convert the Griffith formula of energy balance for propagating cracks

$$\Delta\mathcal{P}^\delta(H) + 2\mathscr{s}\Delta\mathcal{L} \approx 0$$

into the following formula of energy balance

$$\Delta\mathcal{P}_D^\delta + \mathscr{v}\,\Delta\mathcal{V} \approx 0.$$

The latter demonstrates that the asymptotic-variational model (40), (41) in the domain (39) gets a good match with the Griffith criterion of fracture so that shrinkage of the elastic dummy Ω_D^δ due to the damage spread caused by growth of surface microcracks is governed by the Griffith energy criterion, too.

Acknowledgements
This work was supported by the St. Petersburg State University research grant 6.37.671.2013.

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