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ASYMPTOTICAL REGIMES OF THE HYDROFRACTURING PLANAR 3D MODEL

E.V. Shel*

Gazpromneft Science & Technology Center, Saint-Petersburg, Russia

Peter the Great St.Petersburg Polytechnic University (SPbPU), Saint-Petersburg, Russia

*e-mail: Shel.EV@gazpromneft-ntc.ru

Abstract. In the proposed paper the Planar 3D model of hydraulic fracturing was investigated. All of the hydraulic fracturing equations were substituted into the mass conservation law and scaled. Nontrivial dimensionless parameters of the problem were found. Asymptotic regimes of the equation, the Radial or the Pseudo3D model, correspond to the infinitely small or the infinitely large dimensionless parameters. Thus, the limiting conditions of these models were found in the terms of the dimensionless parameters.

Keywords: hydraulic fracturing, geomechanics, Planar3D model, Pseudo3D model, fracture propagation regimes

1. Introduction

At the current time, a various number of mathematical models of hydraulic fracturing are presented in scientific literature, the review on which see, for example, in [1]. Analytical or semi-analytical models, like the Perkins-Kern-Nordgren model [2-3], the Khristianovich-Geertsma-de-Clerk model [4-5], or the Radial model [2,6-7], have a high evaluation speed, but limited application field due to their simplicity. Pseudo3D model, which is, in fact, one-dimensional, is fast enough for practical application, but still not valid for highly heterogenic reservoirs [8-9].

In contrary, the Planar3D model [10-13] is valid for all types of planar fractures, including the fractures in highly heterogenic reservoirs, but has relatively low evaluation speed for practical purposes. Thus, the Planar3D model application should be limited only by the cases which cannot be numerically simulated with the proper accuracy using other existing models. The main parameter for the analytical models applicability is the ratio between fracture length and height. If the ratio is high, the Pseudo3D model is valid, if the ratio is low – the KGD model should be used. In case of homogeneity of the reservoir conditions, we should apply the radial model.

Nevertheless, the existing criterion is based on the parameters, which themselves are not the initial conditions of the problem, but the results of the numerical simulation. Therefore, the conclusion about the model applicability can be made only after numerical modeling, which is not suitable for field applications. In real practice, a hydrofracturing engineer has only a 1D geomechanical model and a pumping schedule. A modeling result, including the length to height ratio, cannot be easily predicted before modeling. For such purposes, we would need a dimensionless condition on the initial parameters. Moreover, since the analytical models were developed before the more general Planar3D model, the connection between those has not been rigorously proven yet. The main problem is that the transition

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between a two-dimensional Planar3D model and one-dimensional Pseudo3D models is not obvious, and, as presented in this work, requires more than the length/height ratio.

In the proposed work, the non-dimensional formulation of the Planar3D model was obtained, and the full set of the dimensionless parameters of the equation was derived. Using a different set of conditions on small dimensionless parameters in the original equations of the Planar3D, the transition into the Pseudo3D model and analytical model was proven. Thus, the Pseudo3D model appeared to be the limit of the Planar3D model in the case of a low ratio between viscous pressure gradient and the rock stress gradient. In contrary, the Radial model is the limit of the Planar3D model in case of a high value of the abovementioned ratio. In the case of point injection source (for the horizontal well) the Planar3D model has the asymptote of the Radial model at the short time of injection.

2. Mathematical formulation of the Planar3D model

The Planar3D model supposes that the fracture surface is planar and perpendicular to minimum rock stress. The containing medium is assumed to be multilayered, with horizontal interfaces between different layers. Each layer is considered as an isotropic and homogeneous medium. These assumptions lead to the elastic equation in each layer:

$$\sigma_{ij} = \sigma_{ij}^0 + \lambda_k \theta \delta_{ij} + 2\mu_k e_{ij}, \tag{1}$$

where λ_k, μ_k are Lame constants of the layer with index k, σ_{ij}^0 is the initial stresses before hydrofracturing.

The fracture width is designated as a jump in the displacement projection, perpendicular to the fracture surface (see Fig. 1). In the area without fractures, according to the symmetry of the problem, the displacement projection perpendicular to the fracture surface equals zero, as well as the shear stress.

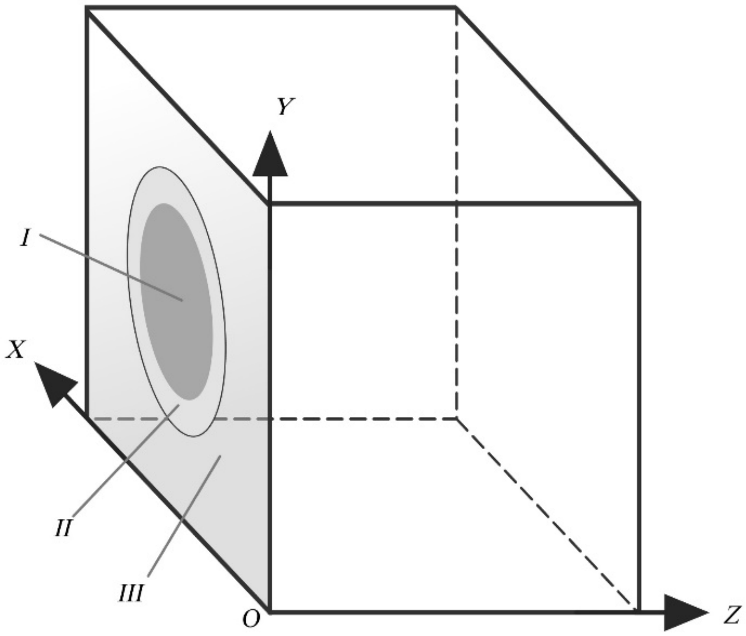

Fig. 1. The geometry of the Planar3D model problem

To sum up, the boundary conditions are as follows:

$$w|_{I+II} = (u_z^+ - u_z^-)|_{I+II}; w|_{III} = 0. \tag{2}$$

From the elasticity problem the normal stress at the fracture surface can be derived, which equals to the fluid pressure in the area of fracture penetration:

$$p|_{I+II} = \sigma_{zz}|_{I+II}. \tag{3}$$

For the elastic media, homogeneous by the elastic constants, the pressure can be obtained through double-layer potential:

$$p(x,y) - \sigma(x,y) = -\frac{E'}{8\pi(1-\nu^2)} \oiint \frac{\nabla_0 w(r_0)(\vec{r_0}-\vec{r})dS_0}{|r_0-r|^3}, \tag{4}$$

where E' is the plain-strain modulus of the elastic medium, $\sigma(x,y)$ are the initial stresses at infinity, perpendicular to the fracture plane. Usually, $\sigma(x,y)$ are considered as linear or constant functions depending only on y in each layer, so in further equations, it can be replaced with $\sigma(y)$.

The mass conservation law of the fluid is constructed in the following form in the Planar3D model:

$$\frac{\partial w}{\partial t} + div\vec{q} + q_l = q_i, \tag{5}$$

where w is the fracture width, defined in Equation (2), $\vec{q}$ is the fluid flux, integrated over the fracture aperture, q_l is the leakage rate of the, q_i – source function, usually approximated as 1- or 2-dimensional Dirac delta-function.

Leakage rate of the fluid in most cases is assumed to be corresponding with the Carter law:

$$q_l = \frac{2C_l}{\sqrt{t-t_0}}. \tag{6}$$

Fluid flux along the fracture can be evaluated by Poiseuille's law for pseudo-plastic fluid rheology [14]:

$$\vec{q}|\vec{q}|^{n-1} = -\frac{1}{\psi k'} w^{2n+1}\nabla p, \tag{7}$$

$$\psi = \frac{2^{n+1}(2n+1)^n}{n^n}. \tag{8}$$

The fracture propagation condition may have quite complicated forms, but all these forms have an equivalent in form of the potential energy variation [15]:

$$\delta A_{frac} = \frac{{K_{Ic}}^2}{E\prime}\delta S = 2\gamma_{frac}\delta S, \tag{9}$$

where K_{Ic} is the medium fracture toughness, δS – fracture surface variation, δA_{fract} – energy variation on fracture growth. Most equations in this section can be seen, for example, in [10].

3. Scaling of the Planar3D model

Scaling of all Planar3D equations is the key feature for asymptotic analyses. Each asymptotic mode can be derived from the original system only when some terms of the equation are proven to be negligible. Thus, a system of equations should have a dimensionless form for such estimations, when all required functions in the equations have the order of 1, and all scaling parameters are taken out as factors that determine the dimensionless parameters of the equations.

In the problem of hydraulic fracturing, there are two required functions: fracture width distribution $w(x,y,t)$ and fluid pressure $p(x,y,t)$, defined on the fracture plane. Fracture form can be defined as the $w(x,y,t)$ function support. Function $p(x,y,t)$ can be derived by the Equation (10) from the function $w(x,y,t)$ hence, to be excluded from the equations.

To obtain the dimensionless form of Equation (4) coordinates x, y should be scaled on some length constant from the geometry of the problem, while fluid pressure should be scaled on some pressure constant from boundary conditions. In an infinite homogeneous medium, such boundary constants do not exist, but in most field appliances this condition is not observed. At least, there is some permeable reservoir with thickness H, minimal rock stress σ_0 and at least two semi-infinite barrier layers with $\Delta\sigma$ rock stress difference with the main reservoir. Thus, a symmetrical 3-layered media is the minimal non-trivial boundary condition for the Planar3D model problem.

For the symmetrical 3-layered media, the scaling is as follows:

$$\tilde{p} = \frac{p_{net}}{\Delta\sigma}; \tilde{x} = \frac{x}{H}; \tilde{y} = \frac{y}{H}; \tilde{w} = w\frac{E'}{\Delta\sigma H}; \tilde{t} = \tilde{V} = \frac{VE'}{\Delta\sigma H^3} = \frac{QE't}{\Delta\sigma H^3}, \tag{10}$$

where V is the injected fluid volume, $p_{net} = p - \sigma_0$ and Q is the constant injection rate.

With such scaling, using the Green formula, Equation (4) transforms as below:

$$\tilde{p} = \frac{1}{8\pi}\iint_{\Omega} \frac{\tilde{w}(\vec{\tilde{r}_0})d\tilde{S}_0}{|\vec{\tilde{r}}-\vec{\tilde{r}_0}|^3}. \tag{11}$$

Using the same scaling in Poiseuille's law, Equation (6) is transformed as:

$$q|q|^{n-1} = -\frac{\Delta\sigma}{\psi k' H}\left(\frac{\Delta\sigma H}{E'}\right)^{2n+1}\vec{\tilde{\nabla}}(\tilde{p}+\tilde{\sigma})\tilde{w}^{2n+1}. \tag{12}$$

Using Scaling (10) and substituting Equations (6,11-12) in the conservation law (5) the following equation can be derived:

$$-\left(\frac{1}{\psi}\right)^{\frac{1}{n}}\left(\vec{\tilde{\nabla}},\vec{\tilde{\nabla}}\left(\frac{1}{8\pi}\iint_{\Omega}\frac{\tilde{w}(\vec{\tilde{r}_0})d\tilde{S}_0}{\left|\vec{\tilde{r}}-\vec{\tilde{r}_0}\right|^3}+\tilde{\sigma}\right)\left|\vec{\tilde{\nabla}}\left(\frac{1}{8\pi}\iint_{\Omega}\frac{\tilde{w}(\vec{\tilde{r}_0})d\tilde{S}_0}{\left|\vec{\tilde{r}}-\vec{\tilde{r}_0}\right|^3}+\tilde{\sigma}\right)\right|^{\frac{1}{n}-1}\tilde{w}^{2+\frac{1}{n}}\right)+\frac{\partial\tilde{w}}{\partial\tilde{t}}\gamma^{\frac{1}{n}}+$$

$$\gamma^{\frac{1}{n}-\frac{1}{2}}\frac{2\tilde{C}}{\sqrt{\tilde{t}-\tilde{t}_0}} = \gamma^{\frac{1}{n}}\delta(\vec{\tilde{r}}). \tag{13}$$

The following combinations of scaling factors were taken as dimensionless parameters:

$$\gamma = \frac{E'^{2n+1}k'Q^n}{H^{3n}\Delta\sigma^{2n+2}}, \tag{14}$$

$$\tilde{C} = C_l\left(\frac{HE'}{Q\Delta\sigma}\right)^{\frac{1}{2}}. \tag{15}$$

Parameter γ represents the ratio between viscous pressure gradient and the gradient of rock stress. As will be shown further, it is the key parameter of the Planar3D model. Parameter $\tilde{C}$ represents the ratio between leakage rate and injection rate.

The last Equation (9), the fracture propagation criteria, can be scaled by dividing by the value of elastic deformation energy $\frac{\Delta\sigma^2 H^3}{E'}$. This scaling will give an additional dimensional equation and a dimensionless parameter:

$$\widetilde{\delta A_{frac}} = \tilde{K}^2\widetilde{\delta S}, \tag{16}$$

$$\tilde{K} = \sqrt{\frac{1}{H}}\frac{K_{Ic}}{\Delta\sigma}. \tag{17}$$

Parameter $\tilde{K}^2$ represents the ratio between the potential energy expended on fracture propagation and the potential energy of the elastic fracture deformation. In practical applications, $\tilde{K}\sim 0.1$, and hence toughness effects are usually neglected. In further work, this assumption is operative, which means that the solutions will be presented in viscosity-dominated mode.

Also, in further work the parameter $\tilde{C}$ is neglected, due to the storage-dominated regime assumption. The assumption of the leakage-dominated regime should be investigated in another research. Most likely, the leak-off dominated regime scaling does not depend on geomechanical parameters and corresponds to the similar scaling of the PKN model.

4. The Radial model asymptote

The first obvious asymptote of the Planar3D model in three-layered media is the case of homogeneous rock stresses. In that setting, if the fracture length and height are well above the perforation cluster (the source of the fluid flow in the model), the problem has an axial symmetry in the fracture plane. Thus, the fracture form will be radial, and the Planar3D model will turn into the radial model. Obviously, for the Radial model in a storage-dominated regime the gradient of the external rock stresses $\vec{\tilde{\nabla}}\tilde{\sigma} = 0$, and the Equation (13) should have the following form:

$$-\left(\frac{1}{\psi}\right)^{\frac{1}{n}}\left(\vec{\tilde{\nabla}},\vec{\tilde{\nabla}}\tilde{p}\left|\vec{\tilde{\nabla}}\tilde{p}\right|^{\frac{1}{n}-1}\tilde{w}^{2+\frac{1}{n}}\right)+\frac{\partial\tilde{w}}{\partial\tilde{t}}\gamma^{\frac{1}{n}}=\gamma^{\frac{1}{n}}\delta(\vec{\tilde{r}}). \quad (18)$$

The γ-parameter (14) tends to infinity, whether the $H\to 0$ or $\Delta\sigma\to 0$. Actually, if $H=0$ or $\Delta\sigma=0$, the problem of three-layered media becomes the problem of a single homogeneous media, and the Equation (13) should transform into the Equation (18) of the Radial problem. Nevertheless, when $H=0$ or $\Delta\sigma=0$, Scaling (10) has no physical meaning. Direct limit of Equation (13) when $\gamma\to\infty$ is meaningless since the term with the highest derivative disappears.

Therefore, Equation (13) should be transformed and rescaled in a form without γ-parameter in every term, except for the term with the $\vec{\tilde{\nabla}}\tilde{\sigma}$, rock stresses gradient, which should be negligible at the $\gamma\to\infty$ limit.

Since the toughness and the leak-off effects were neglected, the Radial model in this assumption has a viscosity-dominated regime scaling, which can be obtained directly from the Equation (18), but for brevity, this scaling (see, for example, [6-7]) will be written explicitly in the abovementioned notation and then substituted directly in the Equations (13,18). For these purposes the fracture width function w and the coordinates on the fracture plain should be rescaled:

$$\vec{\tilde{r}}=\vec{\theta}\gamma^{-\frac{1}{3(n+2)}}\tilde{t}^{\frac{2(n+1)}{3(n+2)}};\ \tilde{p}=p^*(\vec{\theta})\gamma^{\frac{1}{n+2}}\tilde{t}^{-\frac{n}{n+2}};\ \tilde{w}=w^*(\vec{\theta})\gamma^{\frac{2}{3(n+2)}}\tilde{t}^{\frac{2-n}{3(n+2)}}. \quad (19)$$

By substituting this scaling into Equation (18) and dividing by the common factors we obtain the equation in the viscosity-dominated radial regime:

$$-\left(\frac{1}{\psi}\right)^{\frac{1}{n}}\left(\frac{\partial}{\partial\vec{\theta}},w^{*2+\frac{1}{n}}\left|\frac{\partial p^*}{\partial\vec{\theta}}\right|^{\frac{1}{n}-1}\frac{\partial p^*}{\partial\vec{\theta}}\right)+\frac{\partial w^*}{\partial\tilde{t}}\tilde{t}=\delta(\vec{\theta}), \quad (20)$$

$$p^*=\frac{1}{8\pi}\iint_{\Omega}\frac{w^*(\vec{\theta_0})d^2\vec{\theta_0}}{|\vec{\theta}-\vec{\theta_0}|^3}. \quad (21)$$

Equations (20-21) have no dependence on parameter γ, which contained the scaling factors $H,\Delta\sigma$ which cannot be presented in the Radial model. Also, any possible scaling of time retains the form of Equation (20). Equations (20-21) can also be converted back into the dimensional form with (10,19), and the independence of (20,21) on $H,\Delta\sigma$ will be obvious.

Equation (13) for the Planar3D model in three-layered media with the same scaling from (19) turns into the following equation:

$$-\left(\frac{1}{\psi}\right)^{\frac{1}{n}}\left(\frac{\partial}{\partial\vec{\theta}},w^{*2+\frac{1}{n}}\left|\frac{\partial}{\partial\vec{\theta}}\left(p^*+\frac{\tilde{\sigma}}{\gamma^{\frac{1}{n+2}}}\tilde{t}^{\frac{n}{n+2}}\right)\right|^{\frac{1}{n}-1}\frac{\partial}{\partial\vec{\theta}}\left(p^*+\frac{\tilde{\sigma}}{\gamma^{\frac{1}{n+2}}}\tilde{t}^{\frac{n}{n+2}}\right)\right)+\frac{\partial w^*}{\partial\tilde{t}}\tilde{t}=\delta(\vec{\theta}). \quad (22)$$

Condition, which allows using the Radial model Equation (20) instead of the Equation (22) is as follows:

$$\frac{\tilde{\sigma}}{\gamma^{\frac{1}{n+2}}}\tilde{t}^{\frac{n}{n+2}}\ll 1. \quad (23)$$

Given that function $\tilde{\sigma}$ has the order of 1, the (23) can be revised as:

$$\tilde{t}\ll\gamma^{\frac{1}{n}}. \quad (24)$$

This condition means that the higher is the γ, the more is the value of time, at which the Radial model assumption is valid. It means that for low time and point source Radial model of the fracture is always valid. Fracture reaches the size of the initial layer at time $\tilde{t}\sim 1$. If $\gamma\gg 1$, then fracture can exceed the initial reservoir height and remain radial. In practical application the injection time is $\tilde{t}_{inj}>1$. Thus, the condition of the Radial fracture for practical applications can be written simply as:

$$\gamma\gg\tilde{t}_{inj}^{n}>1. \quad (25)$$

Conditions (24-25) depend only on boundary conditions and thus can be used without a direct solution of the Planar3D model. Here we should note that Condition (23) can be applied for every inhomogeneous media with limited rock stress inhomogeneity in order of $\Delta\sigma$, not necessarily the three-layered symmetrical medium since the specific form of rock stress distribution was not used in the derivations above.

5. The Pseudo3D model asymptote

The rigorous mathematical formulation of a cell-based Pseudo3D model with equilibrium height growth can be seen, for example, in [8-9]. The geometry of the problem can be seen in Fig. 2.

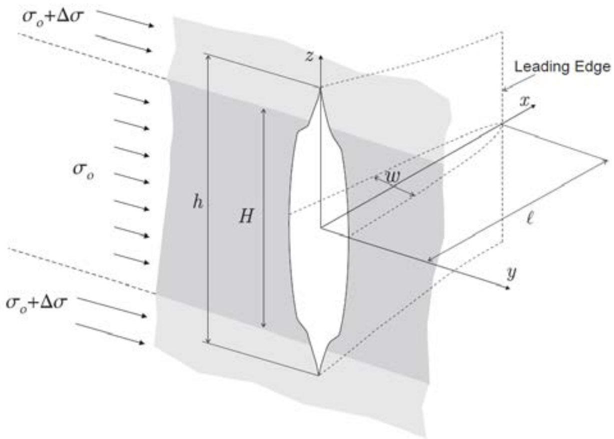

Fig. 2. The geometry of the Pseudo3D model of the hydraulic fracturing (from [8])

The cell-based Pseudo3D model is a model with the assumption of predominate horizontal flow, which allows rewriting the mass-conservation law (5) in the following 1-dimensional form:

$$\frac{\partial Q}{\partial x}+\frac{\partial S}{\partial t}+\frac{2C_l H}{\sqrt{t-t_0(x)}}=0. \qquad (26)$$

Here the Carter leak-offs (6) were used. The 1-dimensional mass-conservation law can be derived by averaging the horizontal flux by the fracture height in accordance with the Poiselle formula (7-8):

$$Q(x)=-\frac{\partial p}{\partial x}\left|\frac{\partial p}{\partial x}\right|^{\frac{1}{n}-1}\left(\frac{1}{\psi k'}\right)^{\frac{1}{n}}\int_{-\frac{h}{2}}^{\frac{h}{2}} w^{2+\frac{1}{n}}dz. \qquad (27)$$

Here the vertical flux was neglected, and thus, the fluid velocity vector is predominantly horizontal. Also, this assumption leads to a lower pressure gradient along with the fracture height, and consequently to almost uniform vertical distribution.

The last assumption used in the Pseudo3D model is the high ratio between fracture length and height. According to these assumptions, the 3-dimensional elasticity Equation (4) is approximated by 2-dimensional elasticity problems on the set of vertical fracture sections. Using the uniform pressure distribution leads to the following equation of the fracture width for a three-layered symmetrical medium:

$$w=\frac{2}{E'}(p-\Delta\sigma)\sqrt{h^2-4z^2}+\frac{4\Delta\sigma}{\pi E'}\left\{\sqrt{h^2-4z^2}\arcsin\left(\frac{H}{h}\right)-z\ln\frac{H\sqrt{h^2-4z^2}+2z\sqrt{h^2-H^2}}{H\sqrt{h^2-4z^2}-2z\sqrt{h^2-H^2}}+\frac{H}{2}\ln\frac{H\sqrt{h^2-4z^2}+\sqrt{h^2-H^2}}{H\sqrt{h^2-4z^2}-\sqrt{h^2-H^2}}\right\}. \qquad (28)$$

Fracture growth is assumed to be quasi-static, therefore using the linear elasticity of fracture mechanics the equilibrium pressure in the fracture vertical profile can be found as:

$$p-\sigma_0=\Delta\sigma\left[1-\frac{2}{\pi}\sin^{-1}\frac{H}{h}\right]+\sqrt{\frac{2}{\pi}}\frac{K_{Ic}}{\sqrt{h}}. \quad (29)$$

If we apply Scaling (10), these equations can be rewritten as (see, for example, [9]):

$$\frac{\partial\tilde{Q}}{\partial\tilde{x}}+\frac{\partial\tilde{S}}{\partial\tilde{V}}+\frac{2\tilde{C}}{\sqrt{\tilde{V}-\tilde{V}_0(\tilde{x})}}=0, \quad (30)$$

$$\tilde{Q}(\tilde{x})=-\frac{1}{\gamma}\frac{\partial\tilde{p}}{\partial\tilde{x}}\left|\frac{\partial\tilde{p}}{\partial\tilde{x}}\right|^{\frac{1}{n}-1}\left(\frac{1}{\psi}\right)^{\frac{1}{n}}\int_{-\frac{\lambda}{2}}^{\frac{\lambda}{2}}\tilde{w}^{2+\frac{1}{n}}d\tilde{z}. \quad (31)$$

Here we used the additional scaling terms on the total horizontal fluid and fracture height at the coordinate:

$$\tilde{Q}(\tilde{x})=\frac{Q(x)}{Q};\lambda=\frac{h(x)}{H}. \quad (32)$$

In addition, there are two boundary conditions:

$$\lambda(L)=1;\tilde{Q}(0)=1. \quad (33)$$

The full solution of the Pseudo3D model can be found in terms of the function that determines the fracture form:

$$\tilde{x}\equiv\tilde{x}(\lambda,\gamma,\tilde{V},\tilde{C},\tilde{K}). \quad (34)$$

It is quite evident that the Pseudo3D model is significally different from the Planar3D model. The main problem that arises from the comparison of the two is the difference in dimensions of elasticity and hydrodynamics equations. While the Planar3D model is, in fact, two-dimensional, the Pseudo3D model is one-dimensional. Therefore, the asymptotic transition of the Planar3D model to the Pseudo3D model is far from obvious. In literature, the key assumption for such transition is the length to height ratio, but as it will be shown further, this assumption is not enough.

The main thesis of this section is that the Planar3D has the Pseudo3D model as an asymptote at the $\gamma \ll 1$ limit.

To prove this statement, we should estimate the pressure gradient inside the fracture. According to the Poiselle law (7), the pressure gradient can be derived as:

$$|\nabla p|=\frac{\psi k' q^n}{w^{2n+1}}. \quad (35)$$

To estimate the maximum limit of pressure gradient along the horizontal x-axes, firstly, we should note that this vector projection is less than the absolute fracture gradient itself. The vertical pressure gradient is proportional to vertical fluid speed, which should be less than horizontal, as higher rock stresses in the barriers reduce vertical fracture growth. Thus, if $\Delta\sigma>0$:

$$\frac{\partial p}{\partial z}\le\frac{\partial p}{\partial x}\le|\nabla p|=\frac{\psi k' q^n}{w^{2n+1}};\Delta\sigma>0. \quad (36)$$

Secondly, the maximum module of the fluid flux vector q should be estimated. In fact, the fluid flux in the fracture will be maximal, if the leakage is neglected and the whole injection rate Q is pointed in one direction.

The next assumption to be used is that the fracture height is at least above the reservoir thickness H. In a reverse case if the fracture has not reached the formation boundary the dimensionless time will be less than $\tilde{t}=\tilde{V}=\frac{VE'}{\Delta\sigma H^3}<1$ and its form will be radial, according to the previous Section 4. In case when $\tilde{t}>1$, the fluid flux in any direction will be less than $\frac{Q}{H}$. The equality may be reached only if the fracture height is equal to the pay thickness H and the whole fluid moves in the horizontal direction. By this means it leads to the following estimation:

$$\tilde{t}>1;\ |\nabla p|=\frac{\psi k' q^n}{w^{2n+1}}\le\frac{\psi k' Q^n}{H^n w^{2n+1}}\sim\frac{k' Q^n {E'}^{2n+1}}{H^{3n+1}\Delta\sigma^{2n+1}}. \quad (37)$$

Here we used the scaling approach in a similar way to Section 3 and Equation (10).
Using this estimation the physical meaning of parameter γ may be explained:

$$\frac{\nabla pH}{\Delta\sigma} \sim \gamma. \tag{38}$$

Thus, the γ parameter by order corresponds to the ratio between the fluid pressure difference in the fracture, which is the part of the net pressure in Equation (4), and the rock stresses contrast.

This estimation may be used in the fracture growth criteria of Griffiths (9). Let the p_0 be the current net pressure at the fracture initiation point. Then, the fracture growth criteria in horizontal direction give the following estimation:

$$(p_0 + \nabla p * x)\sqrt{L} \sim K_{Ic}. \tag{39}$$

Alternatively, in the scaled form:

$$\tilde{p} - \gamma * \tilde{L} \sim \frac{\tilde{K}}{\tilde{L}^{\frac{1}{2}}}. \tag{40}$$

By analogy, using Equation (29) the following estimation of the fracture growth on the z-axis gives:

$$p_0\sqrt{h} - \nabla p h^{\frac{3}{2}} - \Delta\sigma\left(\sqrt{h} - \frac{H}{\sqrt{h}}\right) \sim K_{Ic}, \tag{41}$$

$$\tilde{p} - \gamma\tilde{h} - \left(1 - \frac{1}{\tilde{h}}\right) \sim \frac{\tilde{K}}{\tilde{h}^{\frac{1}{2}}}. \tag{42}$$

Here it should be kept in mind that in practical application dimensionless toughness (17) may be estimated as $\tilde{K} \ll 1$. Only in that case the following estimations are valid:

$$\tilde{p} - \gamma\tilde{L} \sim \frac{\tilde{K}}{\tilde{L}^{\frac{1}{2}}} \sim 0 \Rightarrow \tilde{L} \sim \frac{\tilde{p}}{\gamma}, \tag{43}$$

$$\tilde{p} - \gamma\tilde{h} - \left(1 - \frac{1}{\tilde{h}}\right) \sim \frac{\tilde{K}}{\tilde{h}^{\frac{1}{2}}} \sim 0 \Rightarrow \tilde{h} \sim \frac{1}{1 + \gamma\tilde{h} - \tilde{p}} < \frac{1}{1 - \tilde{p}}. \tag{44}$$

Using those the estimation of fracture height to length ratio may be obtained:

$$\frac{h}{L} = \frac{\tilde{h}}{\tilde{L}} \sim \frac{\gamma}{\tilde{p}(1 - \tilde{p})}. \tag{45}$$

Based on these results, the estimation of the vertical and horizontal flux ratio is as stated below:

$$q_x \sim w * \frac{\partial L}{\partial t};\, q_y \sim w * \frac{\partial h}{\partial t}, \tag{46}$$

$$\frac{q_y}{q_x} \sim \frac{\frac{\partial h}{\partial t}}{\frac{\partial L}{\partial t}} \sim \frac{h}{L} \sim \frac{\gamma}{\tilde{p}(1 - \tilde{p})}. \tag{47}$$

In the Pseudo3D model, this ratio should be small, and thus it should be:

$$\frac{\gamma}{\tilde{p}(1 - \tilde{p})} \ll 1. \tag{48}$$

Noting, that the term $\tilde{p}(1 - \tilde{p})$ has the maximum of ¼, the following condition is obvious:

$$\gamma \ll 1. \tag{49}$$

In addition, there are two additional conditions on the dimensionless net pressure arising from Condition (48):

$$\tilde{p} \gg \gamma, \tag{50}$$

$$(1 - \tilde{p}) \gg \gamma. \tag{51}$$

Condition (50) means that when net pressure is too low, the Pseudo3D model assumptions cannot be implemented. It applies to, for example, the early stages of fracture evolution, when the radial fracture has reached the reservoir boundaries but has not meaningfully grown horizontally or when the fracture breaks into the layer with lower rock stresses.

Condition (51) means that the net pressure should not approach the rock stresses difference. In other cases, a rapid breakthrough will occur and the vertical fluid flux would not be negligible.

All of these assumptions lead to a1-dimensional fluid flow Equation (26), but they do not necessarily lead to the reduction of a 3-dimensionless elasticity problem to a 2-dimensionless one. This statement should be proven separately. To prove that, Equation (4) should be transformed.

On the one hand, Equation (4) may be revised as:

$$p(x,y)-\sigma(x,y)=-\frac{E}{8\pi(1-\nu^2)}\oiint\frac{\nabla w(r_0)(\overrightarrow{r_0}-\vec{r})dS_0}{|r_0-r|^3}. \quad (52)$$

Equation (28), on the other hand, has the following integral equation in its basis [17]

$$p(x,y)-\sigma(x,y)=-\frac{E}{4\pi(1-\nu^2)}\int_a^b\frac{\nabla w(t)}{t-x}dt. \quad (53)$$

To prove the transition from form (52) to the form (53) at the point with the coordinate (x,z), Equation (52) should be rewritten:

$$p(x,y)-\sigma(x,y)=-\frac{E'}{8\pi}\int_{-\frac{h(x)}{2}}^{\frac{h(x)}{2}}dz_0\int_{-L(z)}^{L(z)}dx_0\frac{\frac{\partial w}{\partial z_0}(z_0-z)+\frac{\partial w}{\partial x_0}(x_0-x)}{((z_0-z)^2+(x_0-x)^2)^{\frac{3}{2}}}. \quad (54)$$

Then, we should integrate the term with $\frac{\partial w}{\partial x_0}$ by dx_0 in parts:

$$\int_{-L(z)}^{L(z)}dx_0\frac{\frac{\partial w}{\partial x_0}(x_0-x)}{((z_0-z)^2+(x_0-x)^2)^{\frac{3}{2}}}=\frac{w}{((y_0-y)^2+(x_0-x)^2)^{\frac{3}{2}}}\Big|_{--L(z)}^{L(z)}-3\int_{-L(z)}^{L(z)}dx_0\frac{w(x_0-x)^2}{((y_0-y)^2+(x_0-x)^2)^{\frac{5}{2}}}-\int_{-L(z)}^{L(z)}dx_0\frac{w}{((z_0-z)^2+(x_0-x)^2)^{\frac{3}{2}}}. \quad (55)$$

The boundary terms are equal to zero due to the boundary condition on the fracture opening $w=0$ on the fracture edge. Then, the second term with $\frac{\partial w}{\partial z_0}$ should also be integrated in parts:

$$\int_{-L(z)}^{L(z)}dx_0\frac{\frac{\partial w}{\partial z_0}(z_0-z)}{((z_0-z)^2+(x_0-x)^2)^{\frac{3}{2}}}=\int_{-L(z)}^{L(z)}\frac{\partial w}{\partial z_0}\frac{1}{z_0-z}\frac{\frac{dx_0}{z_0-z}}{\left(1+\frac{(x_0-x)^2}{(z_0-z)^2}\right)^{\frac{3}{2}}}. \quad (56)$$

Alternatively, after hyperbolic substitution:

$$\int_{-L(z)}^{L(z)}dx_0\frac{\frac{\partial w}{\partial z_0}(z_0-z)}{((z_0-z)^2+(x_0-x)^2)^{\frac{3}{2}}}=$$

$$\frac{\partial w}{\partial z_0}\frac{1}{z_0-z}\tanh(\operatorname{asinh}\frac{(x_0-x)}{(z_0-z)})\Big|_{-L(z)}^{L(z)}-\int_{-L(z)}^{L(z)}\frac{\frac{\partial^2 w}{\partial z_0\partial x_0}}{z_0-z}\tanh(\operatorname{asinh}\frac{(x_0-x)}{(z_0-z)})\,dx_0. \quad (57)$$

Exclusive of the leading term with $\frac{\partial w}{\partial z_0}\frac{1}{z_0-z}\sim\frac{w}{h^2}$, all other terms have orders of $\frac{w}{L^2}$ or $\frac{w}{hL}$. If $\frac{h}{L}\ll 1$, these terms are negligible. The leading term resolves itself into the Equation (53) of the 2D elasticity theory only when:

$$\frac{(L(z)-x)}{(z_0-z)}\gg 1. \quad (58)$$

Therefore, this condition will be valid only in points with the distance from the fracture tip of the order of H. Therefore, even under all Conditions from (36,48-51), the Pseudo3D solution would diverge from the exact Planar3D solution in fracture tips and in points of the maximum or minimum heights. Bearing these limitations in mind, the transition of the Planar3D model into the Pseudo3D model under Conditions (48-51) is now proven.

To conclude this section, we should note that by analogy with the Radial asymptote, the Pseudo3D asymptote has its own scaling. Due to the transition of the 3-dimensional theory of elasticity to the 2-dimensional, which was proven above, the x-axis in Pseudo3D equations becomes independent from z-axis. Thus, the Pseudo3D model can also be rescaled as follows:

$$\tilde{x}=\frac{x^*}{\gamma};\tilde{t}=\frac{t^*}{\gamma};\tilde{z}=z^*. \quad (59)$$

All other parameters are scaled according to (10).

Using this specific scaling, Equations (30,31) can be rewritten as (see, for example, [8-9]):

$$\frac{\partial \tilde{Q}}{\partial x^*}+\frac{\partial \tilde{S}}{\partial V^*}+\frac{2\tilde{C}\gamma^{-\frac{1}{2}}}{\sqrt{V^*-V^*{}_0(\tilde{x})}}=0, \tag{60}$$

$$\tilde{Q}(x^*)=-\frac{\partial \tilde{p}}{\partial x^*}\left|\frac{\partial \tilde{p}}{\partial x^*}\right|^{\frac{1}{n}-1}\left(\frac{1}{\psi}\right)^{\frac{1}{n}}\int_{-\frac{\lambda}{2}}^{\frac{\lambda}{2}}\tilde{w}^{2+\frac{1}{n}}d\tilde{z}. \tag{61}$$

From this rescaling, the solution of the Pseudo3D model (34) can be rewritten as:

$$x^*\equiv x^*(\lambda,t^*,\tilde{C}\gamma^{-\frac{1}{2}},\tilde{K}). \tag{62}$$

This solution does not depend on γ directly, but only on the combination of parameters $\lambda, t^*, \tilde{C}\gamma^{-\frac{1}{2}}, \tilde{K}$. The same result was obtained in [8], scaling from this work differs only by numerical factors from scaling (59). Additionally, this exact scaling was used in the previous work [16].

6. Discussion

The foregoing research indicates that the Planar3D model, according to (25), has the Radial viscosity-dominated asymptote at the $\gamma \gg 1$ limit and relatively small times $\tilde{t}^n_{inj} \ll \gamma$, if the $\tilde{K} \ll 1, \tilde{C}=0$. It means that the fracture length and pressure grow according to the power law, see, for example, [6-7], or Equation (19):

$$L=\ C_1 H\gamma^{-\frac{1}{3(n+2)}}\tilde{t}^{\frac{2(n+1)}{3(n+2)}}=C_1(\frac{Q^{n+2}E'}{k'})^{\frac{1}{3(n+2)}}t^{\frac{2(n+1)}{3(n+2)}}, \tag{63}$$

$$p_0-\sigma_0=C_2\Delta\sigma\gamma^{\frac{1}{n+2}}\tilde{t}^{-\frac{n}{n+2}}\ =C_2\left(k'E'^{n+1}\right)^{\frac{1}{n+2}}t^{-\frac{n}{n+2}}. \tag{64}$$

Here L is the fracture half-length and $p_0-\sigma_0$ is the net pressure at the initiation point, C_1, C_2 are constants, which can be evaluated numerically. According to the work [7], for a Newtonian fluid ($n=1$), $C_1\approx\frac{0.6978}{(12)^{1/3}}\approx 0.3048;\ C_2\approx 1.12*(12)^{1/3}\approx 2.56$.

On the contrary, at the limit $\gamma \ll 1$, if the $\tilde{K}\ll 1, \tilde{C}=0$, and at the medial time, when $\tilde{p}(1-\tilde{p})\gg\gamma$, the Pseudo3D asymptote exists. According to Equation (62), and the works [3,8-9], in terms of this work, by analogy with (63), in the Pseudo3D model fracture length and pressure grow according to the power law:

$$L=\ C_3 H\gamma^{-\frac{1}{2n+3}}\tilde{t}^{\frac{2n+2}{2n+3}}=C_3(\frac{Q^{n+2}E'}{k'H^{n+3}})^{\frac{1}{2n+3}}t^{\frac{2n+2}{2n+3}}, \tag{65}$$

$$p_0-\sigma_0=C_4\Delta\sigma\gamma^{\frac{1}{2n+3}}\tilde{t}^{\frac{1}{2n+3}}=C_4\left(\frac{k'E'^{2n+2}Q^{1+n}}{H^{3(n+1)}}\right)^{\frac{1}{2n+3}}t^{\frac{1}{2n+3}}. \tag{66}$$

Here C_3, C_4 are numerical constants, which can be evaluated numerically. According to the work [3], $C_3\approx\frac{0.68}{2^{1/5}}\approx 0.59; C_4\approx\ \frac{2.5}{2^{4/5}}\approx 1.44$

Using the Equation (66) and Conditions (50,51) the time estimation of the Pseudo3D model can be made:

$$\gamma^{2n+2}\ll\tilde{t}\ll\frac{1}{\gamma}. \tag{67}$$

The lower limit can be revised if the condition of the high length to height ratio would be used directly. It means that $L\gg H$, and therefore, using Equation (65):

$$1\ll\ \gamma^{-\frac{1}{2n+3}}\tilde{t}^{\frac{2n+2}{2n+3}}. \tag{68}$$

With this condition combined with Conditions (49,67):

$$\gamma^{\frac{1}{2n+2}}\ll\tilde{t}\ll\frac{1}{\gamma}. \tag{69}$$

In turn, using Scaling (10) and formula (14), it can be rewritten for real time:

$$(\frac{k'Q^nH^{3n+6}}{E'H^{3n}Q^{n+2}})^{\frac{1}{2n+2}}\ll t\ll\frac{H^{3n+3}\Delta\sigma^{2n+3}}{E'^{2n+2}k'Q^{1+n}}. \tag{70}$$

The radial model in these terms according to Condition (24) has the following limit:

$$t \ll \left(\frac{{E'}^{n+1}k'}{\Delta\sigma^{n+2}}\right)^{\frac{1}{n}}. \tag{71}$$

Equations (63,64) mean that on the log-log graph $\tilde{L}$ on $\tilde{V}$, the Pseudo3D and Radial solution will be parallel lines with slopes $\frac{2(n+1)}{3(n+2)}$ and $\frac{2n+2}{2n+3}$ respectively. The Planar3D model solutions with different values of γ parameter are curves, that coincide with the Pseudo3D solution lines at $\gamma \ll 1$ and with the Radial solution lines at $\gamma \gg 1$. In the case of medial γ the transition between the two asymptotes can be observed. These plots in Scaling (10) with the same physical meaning can also be seen in the previous work [18].

This set of numerical solutions of the three-layered symmetrical model can be seen in Fig. 3, the Planar3D solutions conform with the theory described above. Different Planar3D solutions were obtained by variation of the stress contrast $\Delta\sigma$ and fracturing fluid consistency k'. The full set of different solution parameters is given in Table 1.

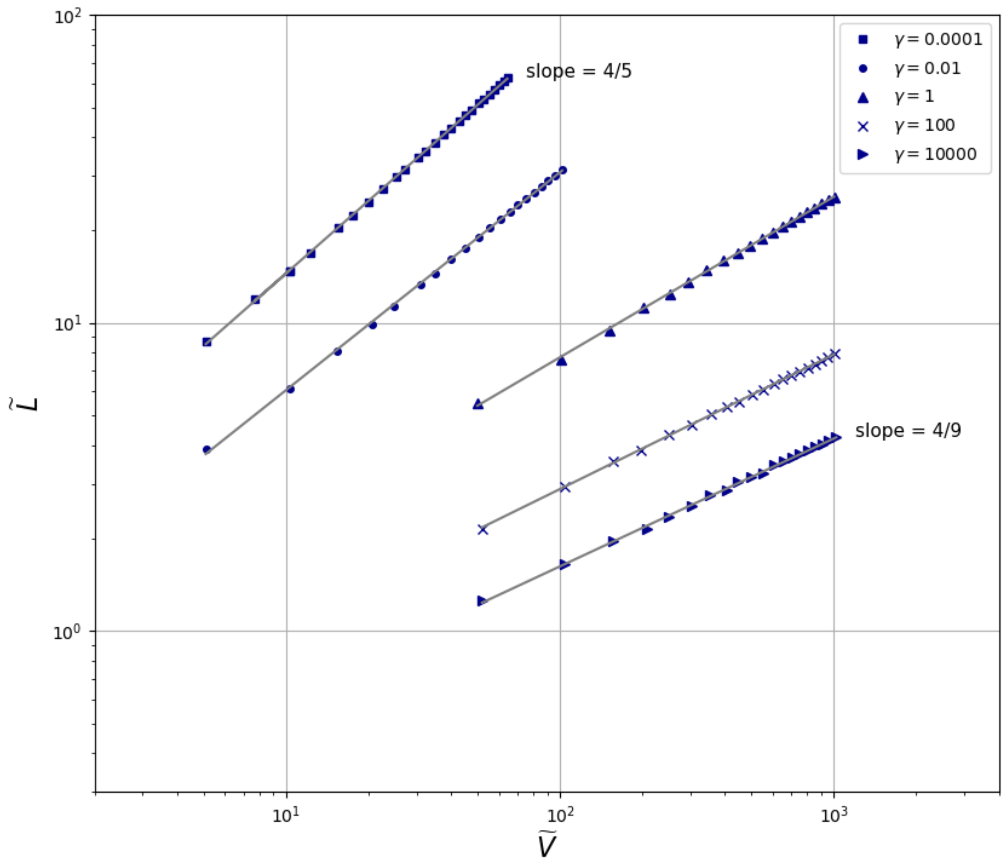

Fig. 3. Dimensionless fracture length $\tilde{L}$ from the dimensionless time or the dimensionless injected volume $\tilde{t} \equiv \tilde{V}$ for different γ-parameters (see Table 1)

Table 1. Parameters for different dimensional parameters γ for a Newtonian fluid with $n = 1$ and slopes of the log-log graphs from the numerical evaluation

H,m	E,GPa	ν	Δσ,MPa	Q,m^3/min	visc, Pa*s	γ	slope
10	10	0	10	6	0.01	1.00E-04	0.78
10	10	0	10	6	1	1.00E-02	0.71
10	10	0	1	6	0.01	1.00E+00	0.52
10	10	0	1	6	1	1.00E+02	0.44
10	100	0	1	6	0.1	1.00E+04	0.44

The upper time limits of the Radial model (24) and the Pseudo3D model (69,70) have close physical meanings. At these time values, both models predict the dimensionless net pressure to be $\tilde{p} \sim 1$. In practice, it means that net pressure is very close to the stress contrast value. The difference is that the Radial model supposes that $\tilde{p} \gg 1$, so that stress jumps can be neglected in the model, while the Pseudo3D model supposes that $\tilde{p} \ll 1$, and therefore the fracture is mainly contained in the initial layers. With the time growing the net pressure in the Radial model decreases according to Equation (64), while in the Pseudo3D model it increases as indicated in Equation (66). Therefore, in both models, dimensionless net pressure tends to the line $\tilde{p} \ll 1$ but from the different sides of the line (see Fig. 4 as an example).

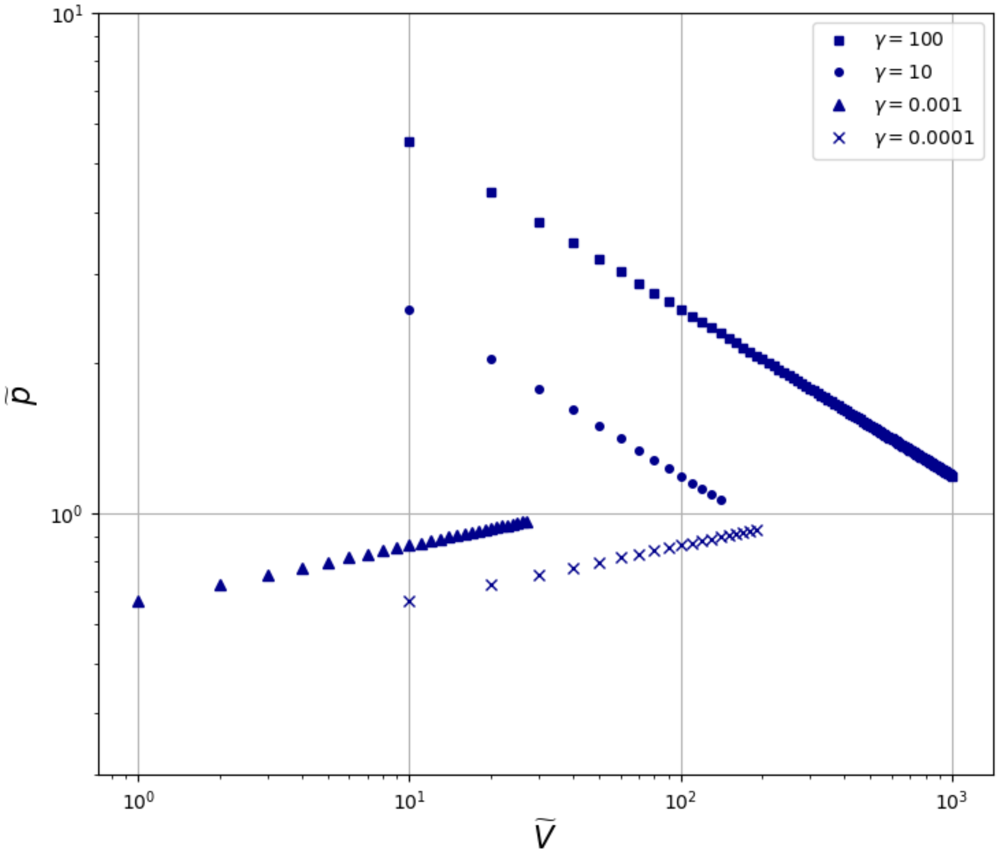

Fig. 4. Dimensionless fracture net pressure $\tilde{p}$ from the dimensionless time or the dimensionless injected volume $\tilde{t} \equiv \tilde{V}$ for different γ-parameters (see Table 1)

7. Conclusion

Using the main equations of the Planar3D model for hydraulic fracturing (1-9), and Scaling (10) proposed in this work, the dimensionless form of Planar3D model equations were obtained for symmetrical three-layered media (13-17). These dimensionless equations were studied under the effect of asymptotic analyses on dimensionless parameters.

The parameter γ (13), obtained for the first time in previous works on the Pseudo3D model and studies designed for engineering appliances [16], appeared to have the key meaning for asymptotic analyses. The γ-parameter has the meaning of the ratio between viscous pressure gradient in the fracture and local rock stress gradient in the reservoir. In the case of the large parameter $\gamma \gg 1$ the asymptotic transition to analytical Radial model [6-7] was proven. This asymptote is valid for high viscosities and low-stress contrasts. At the small parameter $\gamma \ll 1$ the asymptotic transition to the well-known cell-based Pseudo3D model [8-9] was proven. This asymptote corresponds to low viscosities and high-stress contrasts.

Both asymptotic models also have applicability limits on the timescale. These limits were obtained in the dimensionless form (24,69) and in the form of main scaling factors (70,71). The upper time limits have the physical meaning in terms of net pressure – at these critical time values, the net pressure approximately equals the stress contrast. The Pseudo3D also has a limit for minor times. Its physical meaning is simple – fracture length should be greater than fracture height, which is the main assumption of the Pseudo3D model.

The medial values of the γ-parameter Planar3D model shows transitional behavior between two asymptotic regimes, as can be seen in Fig. 3 and Table 1.

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