Identification of Two-Neuron FitzHugh-Nagumo Model Based on the Speed-Gradient and Filtering

A. Rybalko[1, 2] and A. Fradkov[1, 2]
[1)] *Institute for Problems in Mechanical Engineering, RAS, Saint Petersburg, 199178, Russia*
[2)] *Theoretical Cybernetics Department, SPBU, Saint Petersburg, 199034, Russia*

(*Electronic mail: alexandrarybalko21@gmail.com.)

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The paper is devoted to the parameter identification problem for two-neuron FitzHugh-Nagumo models under condition when only one variable, namely the membrane potential, is measured. Another practical assumption is that both variable derivatives cannot be measured. Finally, it is assumed that the sensor measuring the membrane potential is imprecise and all measurements have some unknown scaling factor. This circumstances bring additional difficulties to the parameters estimation problem and, therefore, such case was not studied before. To solve the problem firstly, the model is transformed to a more simple form without unmeasurable variables. Variables obtained from applying second-order real filter-differentiator are used instead of unmeasurable derivatives. Then an adaptive system, parameters of which are estimates of original system parameters is designed. The estimation (identification) goal is to properly adjust parameter estimates. To this end the speed-gradient method is employed. The correctness of the obtained solution is proved theoretically and illustrated by the computer simulation in the Simulink environment. The sufficient conditions of asymptotically correct identification for speed-gradient method with integral objective function are formulated and proved. The novelty of the paper is in that in contrast to existing solutions of the FitzHugh-Nagumo identification problem, we take into account a systematic error of membrane potential measurement. Furthermore, the parameters are estimated for a system composed from two FitzHugh-Nagumo models, which opens perspectives for using the proposed results for modeling and estimation of parameters for neuron population.

The FitzHugh-Nagumo (FHN) model is widely used in many fields of science. This model reflects the basic properties of excitability and originally was obtained to resemble nerve cell dynamics[1,2]. Its generality lets the FHN model also be a model of neuron populations or brain regions[3], cardiac tissue[4], chemical processes[5], etc. Therefore, it is essential to be able to correctly use this model on practice and on this way the identification problem appears. However, straightforward methods of parameters estimation can be inapplicable here, which makes the identification of the FHN model not a trivial task at all. Within an experiment only one variable of the FHN model can be measured and this measurement may include an error because of imperfection of sensors. Fortunately, toolkit of control theory allows to cope with existing difficulties and find simple and elegant solution not only for the single FHN model, but also a system constucted from a few of them. This opens new possibilities in neural activity modeling.

I. INTRODUCTION

In recent years, the use of methods based on mathematical and computer modeling have become widespread in neurobiology[6]. However, the use of mathematical models requires the availability of information about the state and parameters of the dynamical systems that describe the models. This information may not be available in advance because the values of the state variables and parameters of the neuron model may not be known to researchers. The problem becomes even more difficult if it is required to evaluate parameters of a population of neurons or a neural network consisting of a large number of interconnected neurons rather than just one neuron.

In this paper one of systems, which are frequently used in neurobiology, is considered. It is the FitzHugh-Nagumo single neuron model, which is a second-order nonlinear system. The FHN model is a simplification of the Hodgkin-Huxley model and has applications in various fields of science and technology due to its simplicity and generality[5]. In the experiment, only one variable of the FHN model is usually measured – the value of the membrane potential, which creates additional difficulties for the identification and use of this model.

Previously, the problem of estimating parameters of the FHN model was studied by a number of authors. Most of the methods used are stochastic. For example, in (Ref. 7) the maximum likelihood estimation method was applied, while the result in (Ref. 8) was obtained using the Bayesian approach. Many solutions are based on a fairly simple least squares method (Refs. 9–11). In (Ref. 9), the original model was first simplified, which made all subsequent steps easier. Nonlinear filtering for estimating time-varying parameters was used and applied to the FHN model in (Ref. 12).

In this work, we use a new identification method based on cybernetical ideas and tools. The model was extended by introducing an additional dynamical system, which parameters estimate parameters of original system. In contrast to the mentioned works, the proposed deterministic algorithm solves the identification problem for several FHN neuron models and takes into account a systematic error of membrane potential's measurement. In comparison with our previous result[13], an approach based on the speed-gradient method[14] in combination with filtering for tuning system parameters was developed and theoretically substantiated. It opens up prospects for modeling the activity of a population of neurons.

II. ADAPTIVE PARAMETERS ESTIMATOR DESIGN

Let us consider a system composed from two original FitzHugh-Nagumo models:

$$\begin{cases} u_1' = u_1 - \frac{u_1^3}{3} - v_1 + I_{ext}, \\ v_1' = \varepsilon(u_1 - a - bv_1), \\ u_2' = u_2 - \frac{u_2^3}{3} - v_2 + I_{ext}, \\ v_2' = \varepsilon(u_2 - a - bv_2), \end{cases} \tag{1}$$

where $u_1(t), u_2(t)$ are the membrane potentials of the first and the second neurons, $v_1(t), v_2(t)$ are the cumulative effects of all slow ion currents responsible for restoring the resting potential of the nerve cells membranes correspondingly. Parameters a and b determine the conductance characteristics of ion channels, $\varepsilon > 0$ is the relative velocity change in slow ion currents, and I_{ext} is the external current[1,2].

We operate under the assumption that only the membrane potentials without its derivatives are measured and all the parameters are unknown. Furthermore, when measuring the variables u_1 and u_2 from a real neurons some errors will inevitably occur. To take systematic (scaling) errors into account it is assumed that $y_1 = cu_1, y_2 = cu_2$, where c is an apriori unknown scaling factor and $y_1(t), y_2(t)$ are new variables. Following this, let us differentiate the first and the third equations of (1) and then substitute the right hand side of the second and the fourth equations instead of v_1' and v_2' correspondingly. We completely get rid of the unmeasurable variables v_1 and v_2 by expressing them from the first and the third equations:

$$\begin{cases} y_1'' = (1-\varepsilon b)y_1' - \frac{1}{3c^2}(y_1^3)' + \varepsilon(b-1)y_1 - \frac{\varepsilon b}{3c^2}y_1^3 \\ \qquad\qquad + c\varepsilon(a+bI_{ext}), \\ y_2'' = (1-\varepsilon b)y_2' - \frac{1}{3c^2}(y_2^3)' + \varepsilon(b-1)y_2 - \frac{\varepsilon b}{3c^2}y_2^3 \\ \qquad\qquad + c\varepsilon(a+bI_{ext}). \end{cases} \tag{2}$$

Then, we sum two equations of (2) representing the coefficients of the resulting equation as a vector:

$$\theta^* = \begin{pmatrix} 1-\varepsilon b \\ -1/3c^2 \\ \varepsilon(b-1) \\ -\varepsilon b/3c^2 \\ 2c\varepsilon(a+bI_{ext}) \end{pmatrix} \tag{3}$$

and arrive at the equation:

$$(y_1+y_2)'' = \theta_1^*(y_1+y_2)' + \theta_2^*(y_1^3+y_2^3)' + \theta_3^*(y_1+y_2) + \theta_4^*(y_1^3+y_2^3) + \theta_5^*. \tag{4}$$

The coefficients of θ^* represent the set of unknown parameters of (1) in a modified form. We propose to determine these coefficients be means of the speed-gradient method.

In order to design a realistic parameters estimation algorithm we make some actions to remove unmeasurable derivatives of y_1 and y_2 from (4). For this purpose we use an auxiliary filter, specifically a double real differentiator described as follows:

$$\xi = \frac{p^2}{(\tau_1 p+1)(\tau_2 p+1)}\eta, \tag{5}$$

where $\tau_i > 0\, i = 1,2$, $p = d/dt$, $\xi \approx \eta''$. Here the multiplier of η is just the transfer function of the filter. In other words, symbolic relation (5) is equivalent to the differential equation:

$$\tau_1\tau_2\xi(t)'' + (\tau_1+\tau_2)\xi(t)' + \xi(t) = \eta(t)''. \tag{6}$$

In can be easily shown that implementation of the filter (6) does not require differentiation of measured signal η.

Although real differentiators (high pass/low pass filters, dirty differentiators) are well known in control and signal processing[15–17], using them in the state and parameters estimation systems is relatively new. Motivation for the choice of real differentiators consists in the fact that there is always a noise in measurements. This means that exact differentiation is impossible since this will lead to a significant increase in measurement error. On the other hand, a real differentiator can approximately differentiate a signal and filter a noise to some extent. The greater coefficients τ_i, the less accurate (5) differentiates, but also the less a noise is strengthened[18].

Now we apply the filter with the transfer function $W(p) = 1/(\tau_1 p+1)(\tau_2 p+1)$ to both parts of (4) and introduce the filtered variables:

$$\begin{aligned} x_1 &= \frac{p^2}{(\tau_1 p+1)(\tau_2 p+1)}(y_1+y_2), \\ x_2 &= \frac{p}{(\tau_1 p+1)(\tau_2 p+1)}(y_1+y_2), \\ x_3 &= \frac{p}{(\tau_1 p+1)(\tau_2 p+1)}(y_1^3+y_2^3), \\ x_4 &= \frac{1}{(\tau_1 p+1)(\tau_2 p+1)}(y_1+y_2), \\ x_5 &= \frac{1}{(\tau_1 p+1)(\tau_2 p+1)}(y_1^3+y_2^3). \end{aligned} \tag{7}$$

Just note that the second and the third equations of (7) are equivalent to $x_4' = x_2$ and $x_5' = x_3$.

Then the equation (4) transforms to

$$x_1 = \theta_1^* x_2 + \theta_2^* x_3 + \theta_3^* x_4 + \theta_4^* x_5 + \theta_5^* + \Delta, \tag{8}$$

where $\Delta(t)$ is decaying error. In what follows we neglect Δ, since it tends to zero exponentially in view of exponential stability of the filter (5).

The problem is to build an adaptive system providing the output variable estimate $\hat{x}_1(t)$ for the model (8) as the output variable and the neurons' parameter estimates, $\theta_i(t)\ i \in 1:5$, as the parameters, which would provide the identification goal:

$$\begin{aligned} &1)\ \hat{x}_1(t) - x_1(t) \to 0 \text{ for } t \to \infty, \\ &2)\ \theta(t) - \theta^* \to 0 \text{ for } t \to \infty, \end{aligned} \tag{9}$$

where $\theta(t)$ is a column vector of adjustable adaptation parameters $\theta_i(t)$ $i \in 1:5$.

To solve the problem an adaptive system structure is chosen as follows:

$$\hat{x}_1 = \theta_1 x_2 + \theta_2 x_3 + \theta_3 x_4 + \theta_4 x_5 + \theta_5. \tag{10}$$

To design the adaptation loop the speed gradient algorithm is used based on the integral objective function Q_t, which explicitly depends on the adjustable parameters[14]:

$$Q_t = \int_0^t \frac{1}{2}\delta^2(x(s), \theta(s), s)ds, \tag{11}$$

where

$$\begin{aligned}\delta(x,\theta,t) = \hat{x}_1 - x_1 = (\theta_1 - \theta_1^*)x_2 + (\theta_2 - \theta_2^*)x_3 \\ + (\theta_3 - \theta_3^*)x_4 + (\theta_4 - \theta_4^*)x_5 + \theta_5 - \theta_5^*,\end{aligned} \tag{12}$$

is the error between the output variable and its estimate. We arrive at the following adaptation algorithm:

$$\theta' = -\delta \begin{pmatrix} \gamma_1 x_2 \\ \gamma_2 x_3 \\ \gamma_3 x_4 \\ \gamma_4 x_5 \\ \gamma_5 \end{pmatrix}, \tag{13}$$

where γ_i $i \in 1:5$ are positive gains. Finally, the adaptive model is as follows:

$$\begin{cases} y_1'' = \theta_1^* y_1' + \theta_2^*(y_1^3)' + \theta_3^* y_1 + \theta_4^* y_1^3 + \theta_5^*, \\ y_2'' = \theta_1^* y_2' + \theta_2^*(y_2^3)' + \theta_3^* y_2 + \theta_4^* y_2^3 + \theta_5^*, \\ x_2 = \frac{p}{(\tau_1 p+1)(\tau_2 p+1)}(y_1 + y_2), \\ x_3 = \frac{p}{(\tau_1 p+1)(\tau_2 p+1)}(y_1^3 + y_2^3), \\ x_4' = x_2, \\ x_5' = x_3, \\ x_1 = \theta_1^* x_2 + \theta_2^* x_3 + \theta_3^* x_4 + \theta_4^* x_5 + \theta_5^*, \\ \hat{x}_1 = \theta_1 x_2 + \theta_2 x_3 + \theta_3 x_4 + \theta_4 x_5 + \theta_5, \\ \theta_1' = -\gamma_1 \delta x_2, \\ \theta_2' = -\gamma_2 \delta x_3, \\ \theta_3' = -\gamma_3 \delta x_4, \\ \theta_4' = -\gamma_4 \delta x_5, \\ \theta_5' = -\gamma_5 \delta. \end{cases} \tag{14}$$

In (14) first two equations model variables corresponding to membrane potentials. Instead of this equations membrane potentials from real neurons may be used on practice. Other equations in (14) builds estimates for parameters of two neuron model, in other words, they solve the problem. Let us note that if we add more neurons in the (1), then the number of equations in (14), which are responsible for solution, stays the same as well as general structure of the adaptive model. It means that our method potentially can be used for modeling neuron populations without any significant changes.

III. THE ACHIEVEMENT OF THE IDENTIFICATION GOAL

Now let us show that for (14) the identification goal is achieved. For that purpose we formulate the following theorem:

Theorem. Consider the system:

$$x' = F(x, \theta, t), \tag{15}$$

$$\theta' = -\delta(x, \theta, t)\,\Gamma z(x, t), \tag{16}$$

where $z(x(t),t) \in R^{m,1}$ is a vector of observed values and the adjusting algorithm for $\theta(t) \in R^{m,1}$ (16) is obtained by the speed gradient method based on the integral objective function, which speed of changing with respect to the system (15) is $\omega(x,\theta,t) = 0.5\,\delta^2(x,\theta,t)$, where $\delta(x,\theta,t) = \theta(t)^T z(x,t) - \theta^{*T} z(x,t)$ is the difference between some scalar input variable $\hat{x}_1(t) = \theta(t)^T z(x,t)$ and its desired value $x_1(t) = \theta^{*T} z(x,t)$. Assume that these conditions are met:

1. *Smoothness condition.* The functions $F(x,\theta,t)$ and $\nabla_\theta\, \omega(x,\theta,t)$ are continuous in x and θ, piecewise continuous in t and locally bounded uniformly in $t \geqslant 0$, in other words for every $\beta > 0$, there exists a number C such that,

$$\|F(x,\theta,t)\| + \|\nabla_\theta\ \omega(x,\theta,t)\| \leqslant C \tag{17}$$

for $\|x\| \leqslant \beta, \|\theta\| \leqslant \beta, t \geqslant 0$.

2. *Convexity condition.* The function $\omega(x,\theta,t)$ is convex in $\theta \in R^{m,1}$.

3. *Weak reachability condition.* There exists a constant vector $\hat{\theta} \in R^{m,1}$ and a scalar function $\mu(t)$ with the properties

$$\int_0^\infty \mu(t)\ dt < \infty,\ \ \lim_{t\to\infty} \mu(t) = 0. \tag{18}$$

such that for any solution $x(t)$ of (15) the following inequality

$$\omega(x(t), \hat{\theta}, t) \leqslant \mu(t), \tag{19}$$

holds for all $t \geqslant 0$.

4. *Persistent excitation condition.* The vector function $z(x(t),t)$ is persistently excited, in other words this function is bounded for $t \geqslant 0$ and there exists positive numbers L, α, t_0 such that, for every $t > t_0$

$$M_L := \int_t^{t+L} z(x(s),s)z(x(s),s)^T ds \geqslant \alpha I_m. \tag{20}$$

Then, in the system (15), (16) the identification goal (9) is achieved for the integral objective function

$$Q_t = \int_0^t\ \omega(x(s), \theta(s), s)ds. \tag{21}$$

Remark. The persistent excitation condition (20) means, in essence, that the matrix M_L is positive definite on the integration interval $[t, t+L]$. It follows that rows of $z(x(t),t)$ do not approach to any hyperplane in R^m for $t \to \infty$ simultaneously[14].

Proof. First three conditions ensure the achievement of the first part of the identification goal (9): $\hat{x}_1(t) - x_1(t) \to 0$ for $t \to \infty$, and were taken from a theorem, which were formulated and proven in (Ref. 14). Let us use a lemma from (Ref. 19) to prove the second part of the identification goal (9) achievement.

Lemma. Consider the smooth vector function $w(t) \in R^{m,1}$ and the persistently excited function $z(t) \in R^{m,1}$, which is defined on $[0,\infty)$. If $w'(t) \to 0$ and $w(t)^T z(t) \to 0$ for $t \to \infty$, then

$$\lim_{t\to\infty} w(t) = 0. \tag{22}$$

$w(t)^T = (\theta_1 - \theta_1^* \quad \theta_2 - \theta_2^* \quad \theta_3 - \theta_3^* \quad \theta_4 - \theta_4^* \quad \theta_5 - \theta_5^*)$ in our case. From 1-3 we have that $w(t)^T z(t) = \theta(t)^T z(x,t) - \theta^{*T} z(x,t) = \hat{x}_1(t) - x_1(t) \to 0$ for $t \to \infty$; $w'(t) = \theta'(t) = -\delta(x,\theta,t)\,\Gamma z(x,t) = -(\hat{x}_1(t) - x_1(t))\,\Gamma z(x,t) \to 0$ for $t \to \infty$ because of the smoothness condition. Thus, lemma conditions are achieved and, therefore, $\theta(t) - \theta^* \to 0$ for $t \to \infty$. □

Now we have to check if the system (14) satisfies the theorem conditions. Let us reduce (14) to the form (15), (16):

$$\begin{cases} y_1' = y_3, \\ y_2' = y_4, \\ y_3' = \theta_1^* y_3 + 3\theta_2^* y_1^2 y_3 + \theta_3^* y_1 + \theta_4^* y_1^3 + \frac{1}{2}\theta_5^*, \\ y_4' = \theta_1^* y_4 + 3\theta_2^* y_2^2 y_4 + \theta_3^* y_2 + \theta_4^* y_2^3 + \frac{1}{2}\theta_5^*, \\ x_2' = -Ax_2 - Bx_4 + B(y_1 + y_2), \\ x_3' = -Ax_3 - Bx_5 + B(y_1^3 + y_2^3), \\ x_4' = x_2, \\ x_5' = x_3, \\ \theta_1' = -\gamma_1 \delta x_2, \\ \theta_2' = -\gamma_2 \delta x_3, \\ \theta_3' = -\gamma_3 \delta x_4, \\ \theta_4' = -\gamma_4 \delta x_5, \\ \theta_5' = -\gamma_5 \delta, \end{cases} \tag{23}$$

where $A = (\tau_1 + \tau_2)/(\tau_1 \tau_2)$, $B = 1/(\tau_1 \tau_2)$.

1. Let $\|x\| \leqslant \beta, \|\theta\| \leqslant \beta$.
$\|F(x,\theta,t)\| = (y_3^2 + y_4^2 + (\theta_1^* y_3 + 3\theta_2^* y_1^2 y_3 + \theta_3^* y_1 + \theta_4^* y_1^3 + \frac{1}{2}\theta_5^*)^2 + (\theta_1^* y_4 + 3\theta_2^* y_2^2 y_4 + \theta_3^* y_2 + \theta_4^* y_2^3 + \frac{1}{2}\theta_5^*)^2 + (B(y_1 + y_2) - Ax_2 - Bx_5)^2 + (B(y_1^3 + y_2^3) - Ax_3 - Bx_6)^2 + x_2^2 + x_3^2)^{\frac{1}{2}}$; $\nabla_\theta\, \omega(x,\theta,t) = -\delta(x,\theta,t)\,(x_2 \quad x_3 \quad x_4 \quad x_5 \quad 1)^{\mathrm{T}}$, therefore, $\|\nabla_\theta\, \omega(x,\theta,t)\| = |\delta(x,\theta,t)|\,\|(x_2 \quad x_3 \quad x_4 \quad x_5 \quad 1)^{\mathrm{T}}\|$. It is clear that there exists a constant C, which majorizes the of these two expressions, in other words the smoothness condition is achieved.

2. Let us consider the matrix of second derivatives of $\omega(x,\theta,t)$ to prove this function's convexity:

$$\frac{\partial^2 \omega(x,\theta,t)}{\partial\theta_i\,\partial\theta_j} = \begin{pmatrix} x_2^2 & x_2x_3 & x_2x_4 & x_2x_5 & x_2 \\ x_2x_3 & x_3^2 & x_3x_4 & x_3x_5 & x_3 \\ x_2x_4 & x_3x_4 & x_4^2 & x_4x_5 & x_4 \\ x_2x_5 & x_3x_5 & x_4x_5 & x_5^2 & x_5 \\ x_2 & x_3 & x_4 & x_5 & 1 \end{pmatrix} \tag{24}$$

Convexity of $\omega(x,\theta,t)$ is equivalent to this matrix's positive semidefiniteness, which, in turn, is equivalent to nonnegativity of its eigenvalues in our case because this matrix is hermitian. Let us write eigenvalues: $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$, $\lambda_5 = x_2^2 + x_3^2 + x_4^2 + x_5^2 + 1 > 0$. Therefore, the convexity condition is achieved.

3. Let $\mu(t) \equiv 0$. Then, for $\hat{\theta} = \theta^*$ $\omega(x,\hat{\theta},t) \equiv 0$, thus, the weak reachability condition is achieved.

4. In our case $z(x(t),t) = (x_2 \quad x_3 \quad x_4 \quad x_5 \quad 1)^T$. The check of the persistent excitation condition was performed numerically in the Simulink environment. As a result of numerical experiments for some set of initial data and parameters (for details see (25) and (26) in Section IV), we found out that there exists some number L ($\geqslant 6$), $t_0 > 0$ such that, the matrix M_L from the left part of (20) is positive definite, which is shown on FIG. 1a. Because of non-negativity of the integrand in (20) with increase in L positive definiteness will persist due to integral's additivity. Moreover, the minimal eigenvalue of matrix M_L almost linearly depends on L (see FIG. 1b) and this eigenvalue positiveness ensures exponential decrease in the estimation errors (see FIG. 2b). Therefore, the persistent excitation condition is achieved for the small enough number α.

Hence, all the theorem conditions are fulfilled, which means that the identification goal is achieved.

IV. SIMULATION RESULTS

In this section we consider the question of dependence between time of identification goal achievement or its accuracy and initial data and parameter values of system (14). For this purpose we did a few experiments in the Simulink environment. First of all, the graphs of estimation errors (FIG. 2) were obtained for a set of initial data:

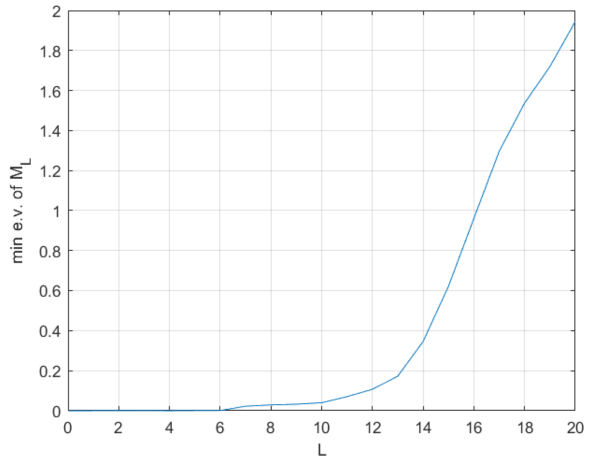
$$\begin{aligned} &y_1(0) = 0.1,\, y_1'(0) = 0.5,\, y_2(0) = 0.45,\, y_2'(0) = 0.2, \\ &x_4(0) = x_5(0) = 1,\, \theta_1(0) = 0.3,\, \theta_2(0) = 0.9, \\ &\theta_3(0) = -0.25,\, \theta_4(0) = 1,\, \theta_5(0) = -0.1; \end{aligned} \tag{25}$$

and parameter values[5]:

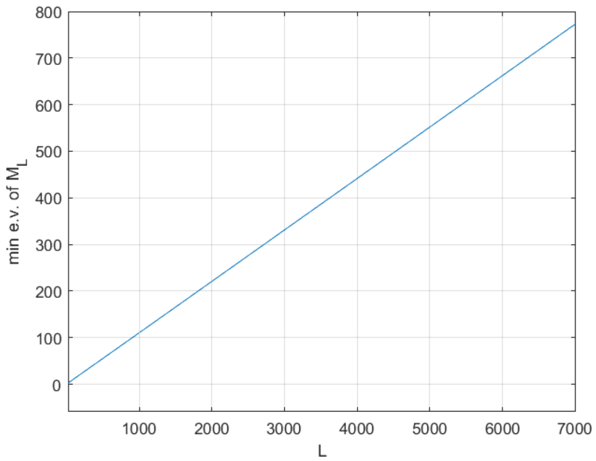
$$\begin{aligned} &I_{ext} = 0.5,\, a = -0.7,\, b = 0.8,\, \varepsilon = \frac{1}{12.5},\, c = 1, \\ &\tau_1 = \tau_2 = 0.01,\, \gamma_i = 1 \;\; i \in 1:5 \end{aligned} \tag{26}$$

The graphs show the identification goal achievement, which was theoretically proved for the state estimate and numerically predicted for the parameters estimates. Moreover, an exponential envelope $\approx \pm 1.44\mathrm{e}^{-0.0022t}$ was built for graphs of the parameters estimates errors. This envelope estimates the convergence rate of the proposed method for chosen initial data (25) and parameters (26). Let us note that although the identification goal is achieved quite fast, which confirms the effectiveness of the proposed approach, this process requires more time than in case of the adaptive parameters estimation of one FHN model[13].

Now let us change initial data and remain the same parameters. FIG. 3 illustrates the graphs of parameters estimations

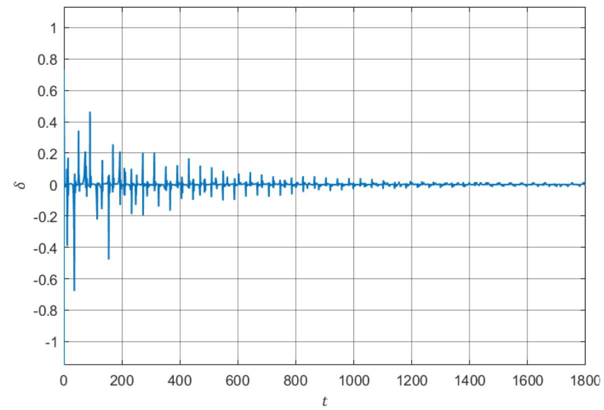


a)

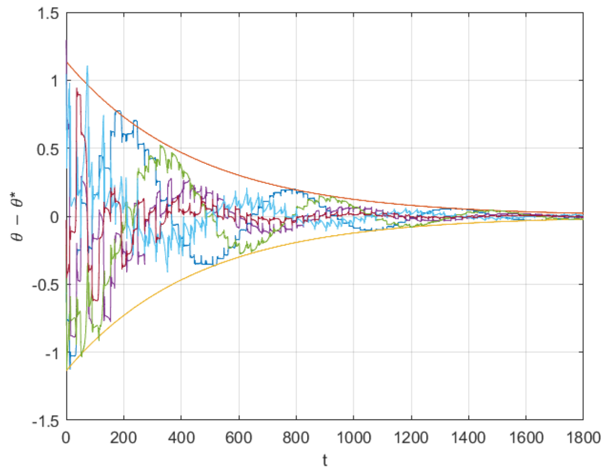


b)

FIG. 1. Graphs of the dependence between the integration interval length L and the smallest eigenvalue of the considered matrix M_L on intervals a) [0, 20], b) [0, 7000].



a)



b)

FIG. 2. Graphs of a) the difference between the values of the output variables $\hat{x}_1$ and x_1; b) the difference between the values of parameters θ_i and theirs estimates θ_i^*, $i \in 1:5$ with the envelope $\approx \pm 1.44\mathrm{e}^{-0.0022t}$ for the set of initial data (25) and parameter values (26).

errors in case of doubled values of (25). It can be clearly seen that this change leads to approximately twofold increase in the convergence time compared to the previous case.

FIG. 4 shows simulation results for initial data (25) and modified parameters values: part of parameters, which are estimated, $(I_{ext}, a, b, \varepsilon, c)$ differs by 20% from (26) and other technical parameters (τ_1, τ_2, $\gamma_i = 1, i \in 1:5$) are the same as in (26). It can be noted that the convergence rate of parameters and theirs accuracy remained almost the same compared to FIG. 2.

As for influence of only the systematic error of membrane potential measurement, c, on the convergence time of the proposed method, there is some correlation as it can be seen from FIG. 5. FIG. 5a illustrates graphs of estimation errors when c is different by 20% and initial data and other parameters are same as in (25) and (26). It can be noticed that the convergence time of parameters estimation rose compared to FIG. 2. FIG. 5b shows that the convergence rate becomes even slower when the error of membrane potential measurement is 60%.

To sum up, the time of identification goal achievement and its accuracy depend from initial data and parameter values. In cases when the convergence rate is slow or results is not accurate enough additional tuning of gains γ_i, $i \in 1:5$ can be done.

V. CONCLUSION

The obtained results show that the design of an adaptive parameter estimation system for two FHN neuron models and the application of the speed gradient method to this system is an efficient way to determine the parameter values in the situation of the immeasurability of both derivatives of the membrane potential and the cumulative action of slow ionic currents. To cope with non-measurable derivatives, differentiating filters were used.

The theoretical contribution of the paper is based on the

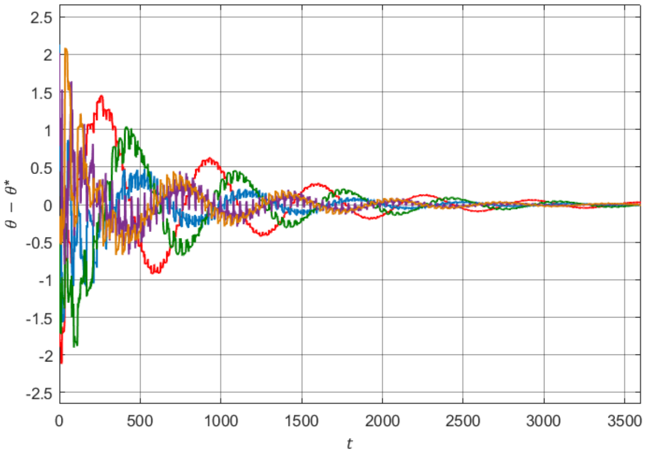


FIG. 3. Graphs o the difference between the values of parameters θ_i and theirs estimates θ_i^*, $i \in 1:5$ for the set of initial data, which is a doubled values of (25), and parameter values (26).

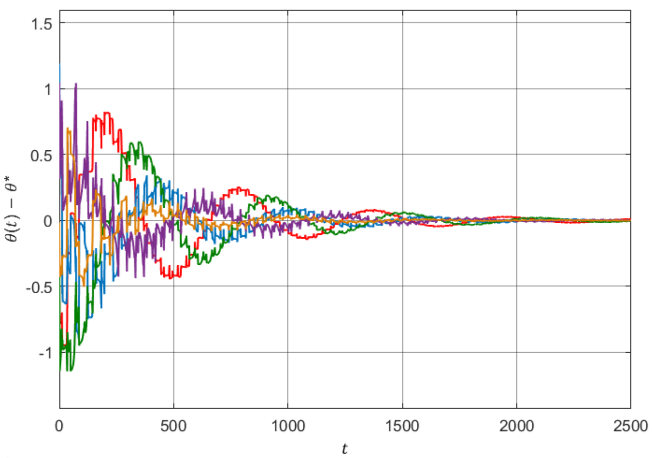
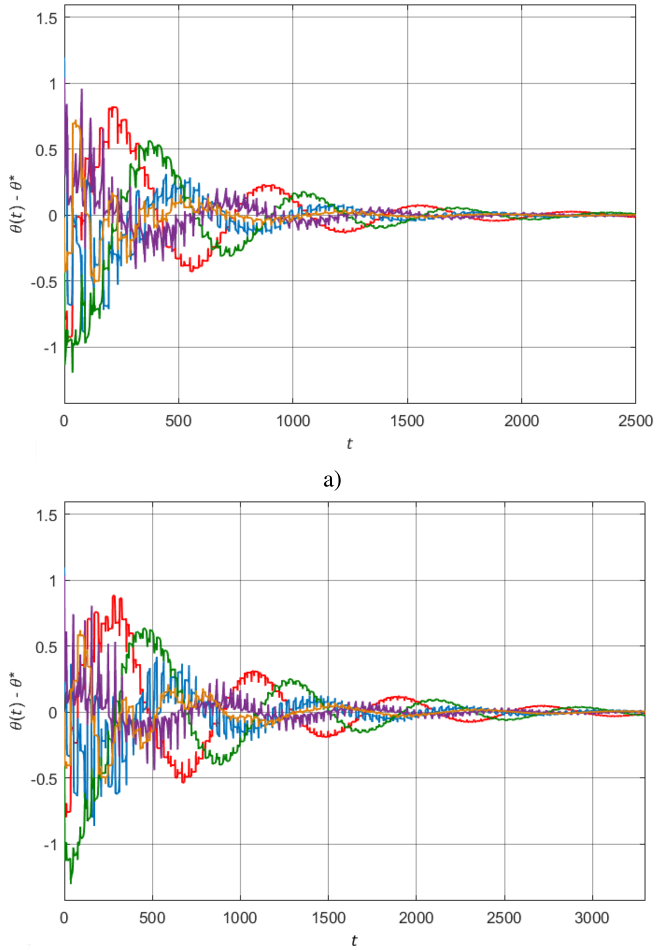


FIG. 4. Graphs of the difference between the values of parameters θ_i and theirs estimations θ_i^*, $i \in 1:5$ for the set of initial data (25) and parameter values, which are different by 20% from (26).

theorem, which provides sufficient conditions for identifying features for the speed gradient algorithms based on the integral objective function and was formulated and proved for the first time. Then it was applied to the identification problem for FHN models.

The computer simulation shows that the adaptive system convergence rate depends on the initial data and parameter values, but anyways, it is rather fast. Particularly, the bigger the membrane potential measurement error, the bigger the convergence time. However, not too large errors do not significantly decrease the convergence rate. Also it is interesting that the convergence rate for two FHN neuron model is slower than in case of one FHN neuron, which means that the more complex the model, the more slow may be its tuning. The additional tuning of algorithm's gains can be done to make the convergence faster anyway.



b)

FIG. 5. Graphs of the difference between the values of parameters θ_i and theirs estimations θ_i^*, $i \in 1:5$ for the set of initial data (25) and parameter values (26) with the error of membrane potential measurement a) 20% b) 60%.

The key achievements of this paper are an accounting of systematic error occurring in the measurement of membrane potential and a mathematically justified algorithm for parameter estimation of the set of two neurons. The proposed approach may provide a new way for the modeling of neuron population and networks and, in future, for the whole brain modeling.

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DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

AUTHOR DECLARATIONS

The authors have no conflicts to disclose.

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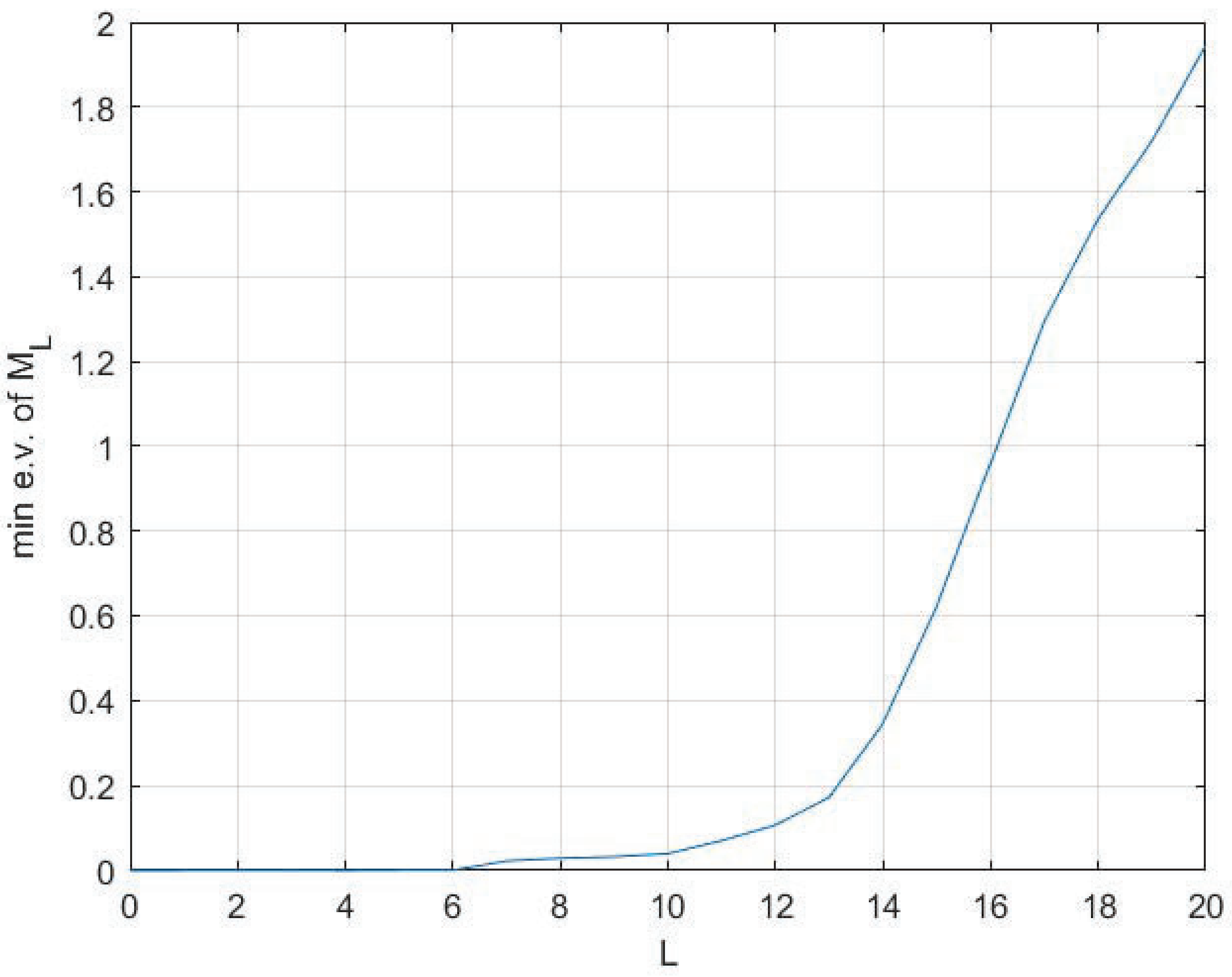
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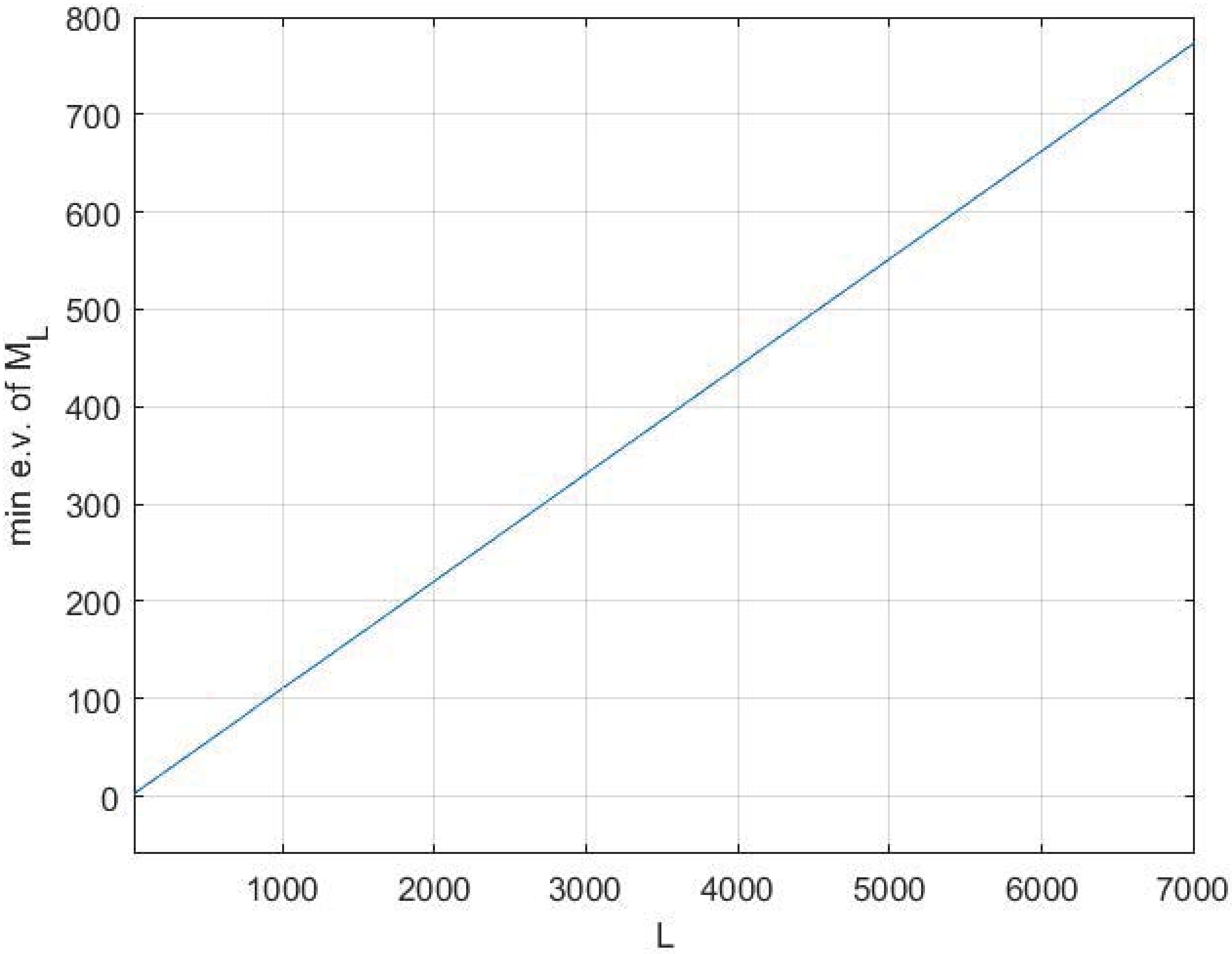
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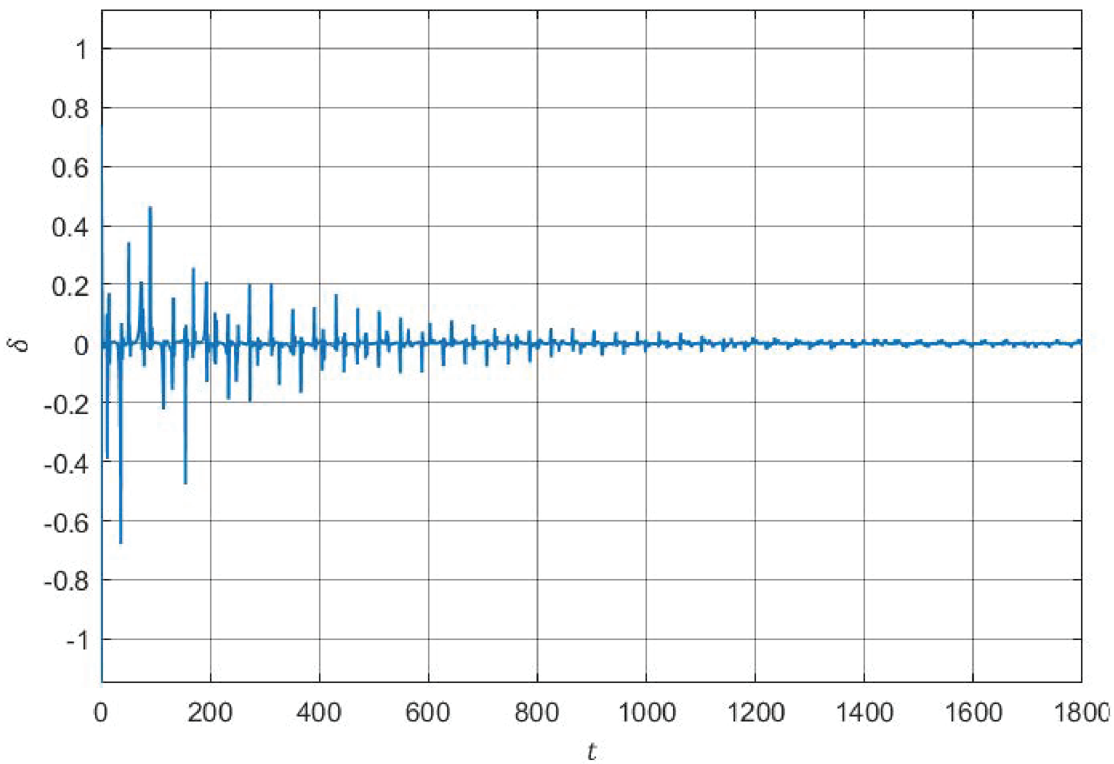
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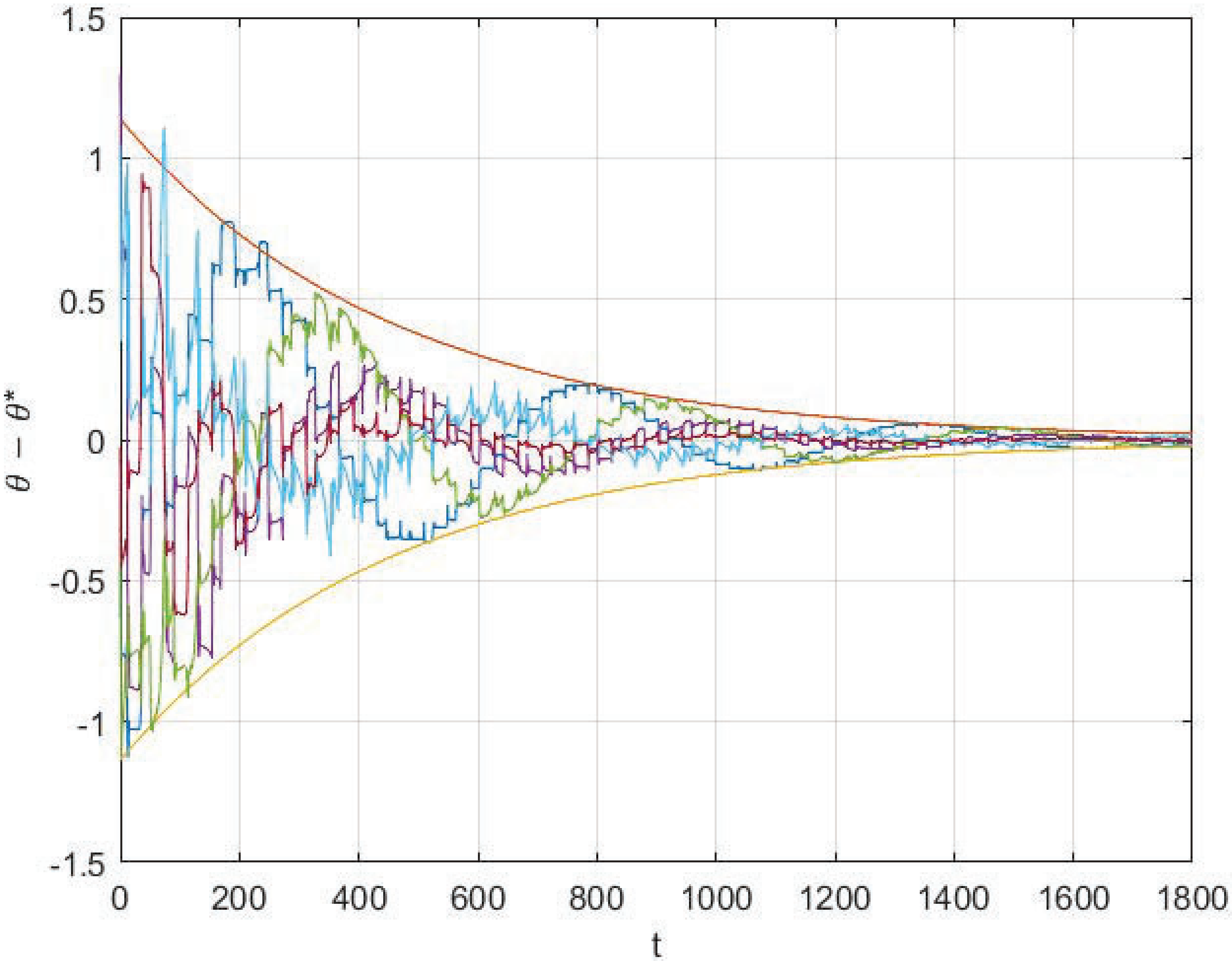
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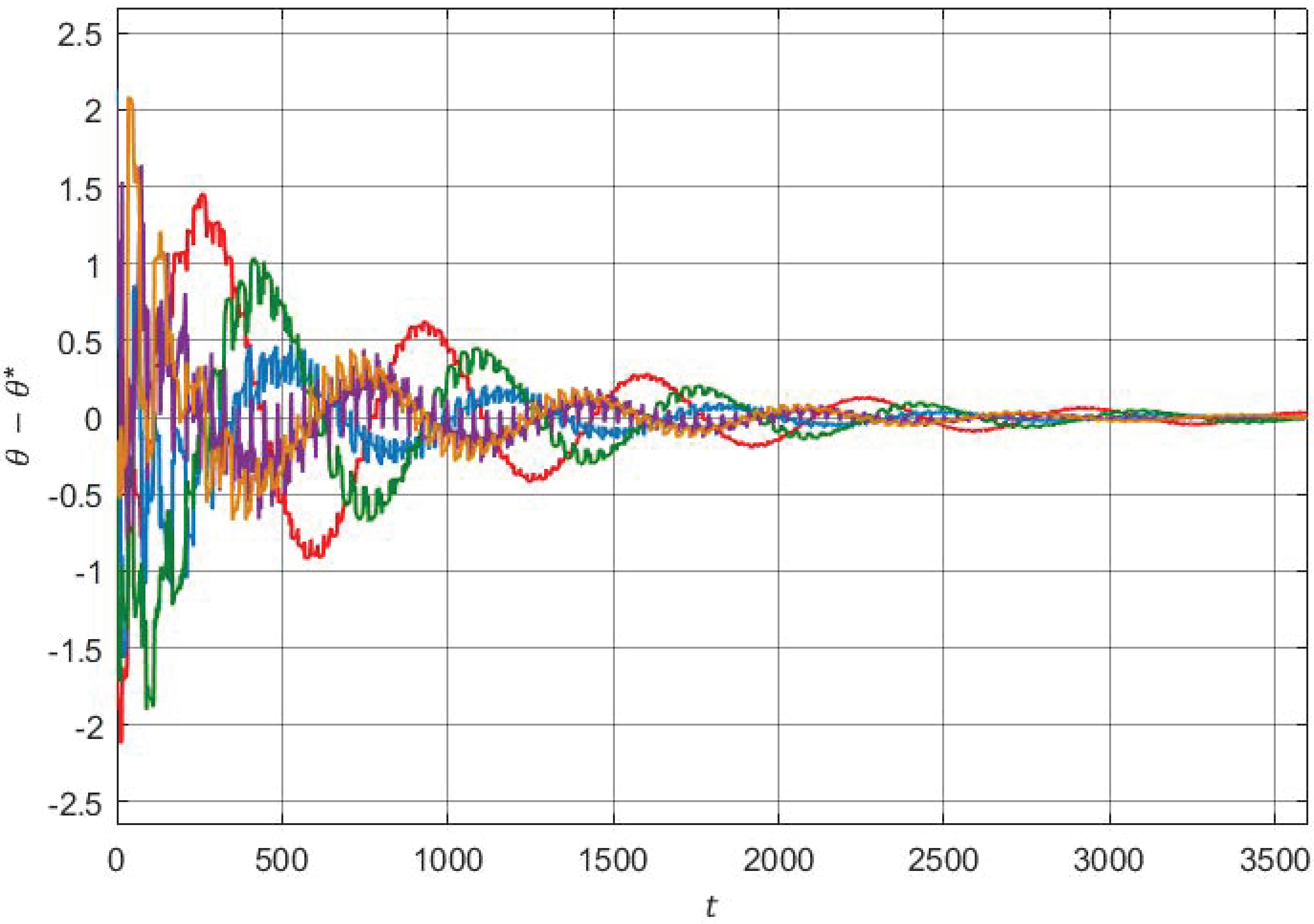
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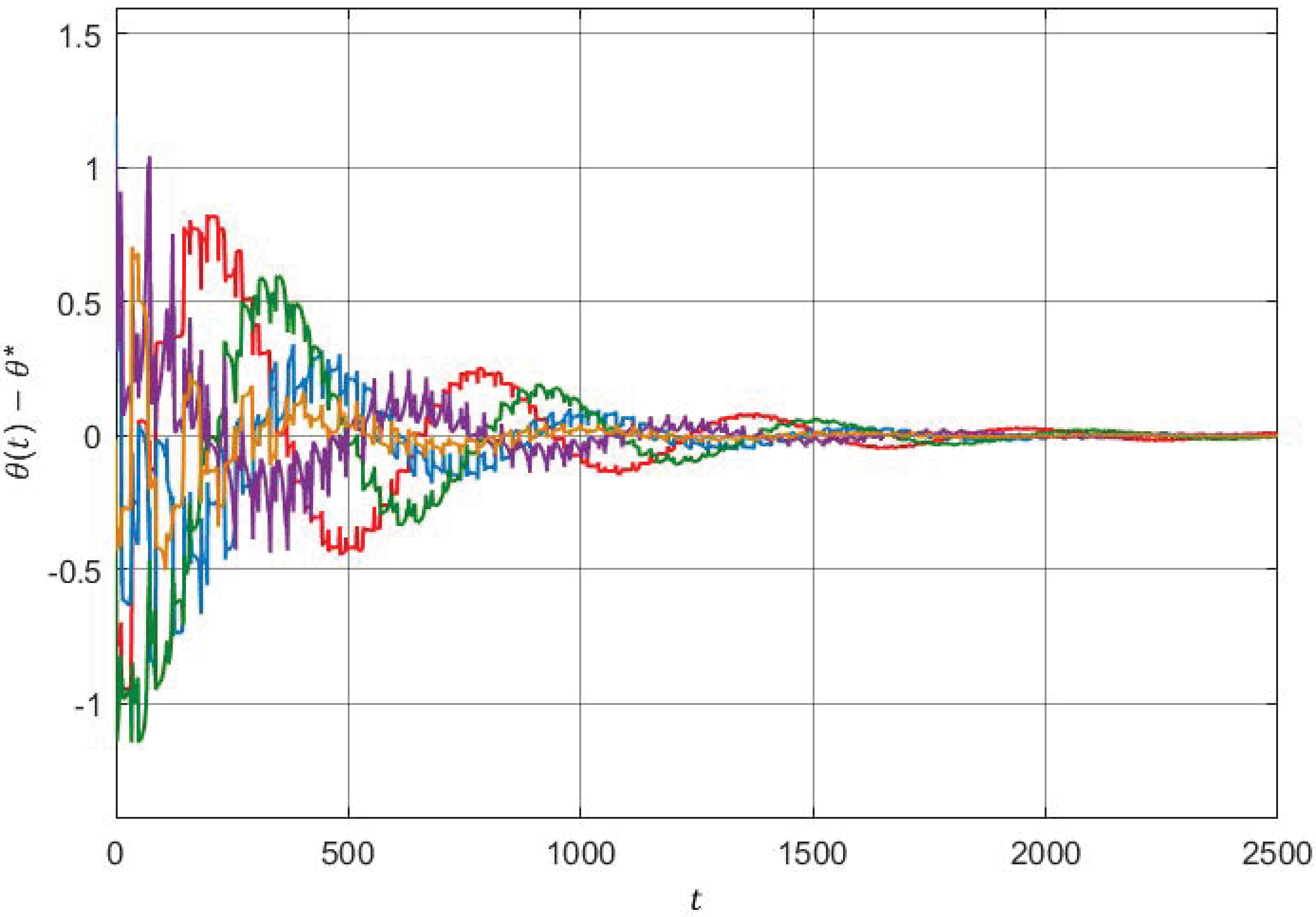
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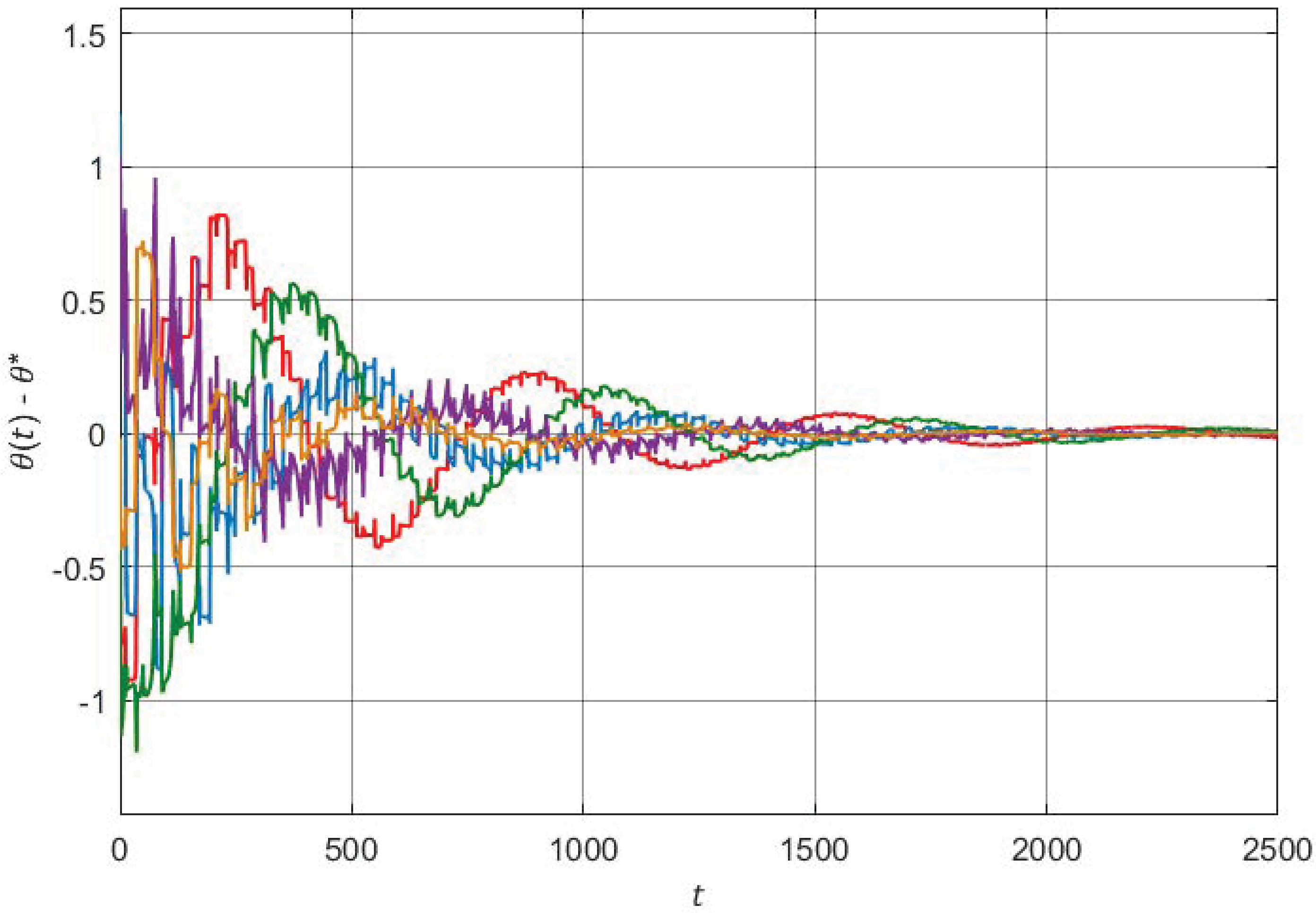
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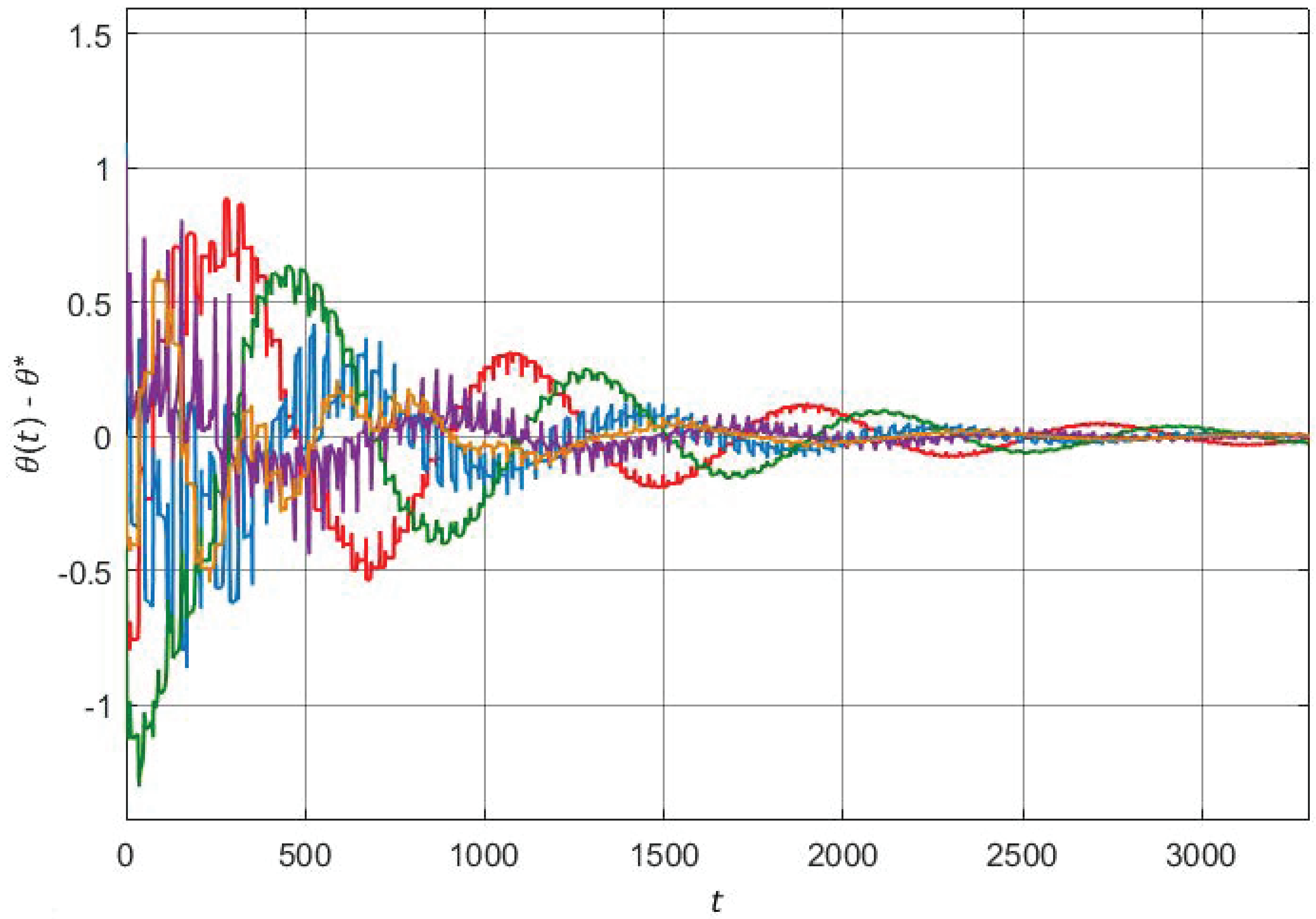
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